TYPES ON STABLE BANACH SPACES

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ABSTRACT. We prove a geometric characterization of Banach space stability. We show that a Banach space X is stable if and only if the following condition holds. Whenever $\hat{X}$ is an ultrapower of X and B is a ball in $\hat{X}$, the intersection $B \cap X$ can be uniformly approximated by finite unions and intersections of balls in X; furthermore, the radius of these balls can be taken arbitrarily close to the radius of B, and the norm of their centers arbitrarily close to the norm of the center of B.

The preceding condition can be rephrased without any reference to ultrapowers, in the language of types, as follows. Whenever τ is a type of X, the set $\tau^{-1}[0, r]$ can be uniformly approximated by finite unions and intersections of balls in X; furthermore, the radius of these balls can be taken arbitrarily close to r, and the norm of their centers arbitrarily close to $\tau(0)$.

We also provide a geometric characterization of the real-valued functions which satisfy the above condition.

1. INTRODUCTION

A separable Banach space X is *stable* if whenever (a_m) and (b_n) are bounded sequences in X and $\mathcal{U}, \mathcal{V}$ are ultrafilters on $\mathbb{N}$,

$$\lim_{\mathcal{U},m} \lim_{\mathcal{V},n} \|a_m + b_n\| = \lim_{\mathcal{V},n} \lim_{\mathcal{U},m} \|a_m + b_n\|.$$

This concept was introduced by J.-L. Krivine and B. Maurey in [5], where the authors proved that every stable Banach space contains almost isometric copies of ℓ_p, for some $1 \leq p < \infty$. This generalized a result of D. Aldous [1] about subspaces of L_1.

The concept of *type* on a Banach space was introduced in [5] as well. If X is a Banach space and $a \in X$, the *type realized by* a is the function $\tau_a \colon X \to \mathbb{R}$ defined by $\tau_a(x) = \|x + a\|$. The *space of types of* X, denoted $\mathcal{T}(X)$, is the closure of $\{\tau_a \mid a \in X\}$ in $\mathbb{R}^X$ with respect to the product topology. The *norm* of a type τ is $\tau(0)$.

The role played by types in [5] generalizes that played by random measures in [1]

Since [5], stable Banach spaces and types have been studied intensely. For a self contained exposition, we refer the reader to [2].

Types can be viewed quite naturally in terms of Banach space ultrapowers as follows. A type on X is a function $\tau \colon X \to \mathbb{R}$ such that there exists an ultrapower $\hat{X}$ of X and an element $a \in \hat{X}$ with

$$\tau(x) = \|x + a\|, \qquad \text{for every } x \in X.$$

In this case, we will say that *a realizes τ in $\hat{X}$*.

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Let X be a normed space. If the type τ is realized in X, say, if $\tau = \tau_a$, then for any $r > 0$, the set

$$\tau^{-1}[0, r] \tag{1}$$

is the ball $\{ x \in X \mid \|x + a\| \leq r \}$. Now, if τ is realized by an element $a \in \hat{X}$, where $\hat{X}$ is an ultrapower of X, the set (1) is the intersection of X with the ball $\{ x \in \hat{X} \mid \|x + a\| \leq r \}$. It is then natural to ask whether (1) can be approximated by balls in X; if so, it is also natural to ask whether the radius of these balls can be taken to be r, and even whether the norm of their centers can be taken to be $\tau(0)$. In this paper we show that all of these approximation properties in fact characterize Banach space stability.

Let X be a normed space. If $\tau \in \mathcal{T}(X)$, let us say that τ is *approximable* if for every $r > 0$ and every $\epsilon > 0$, the set $\tau^{-1}[0, r]$ is within ϵ of a set formed by finite unions and intersections of balls in X. (See Definition 2.2.) Let us say that τ is *strongly approximable* if τ is approximable and the radii of the balls approximating $\tau^{-1}[0, r]$ can be taken arbitrarily close to r, and the norm of their centers arbitrarily close to the norm of τ. In Theorem 4.1, we prove that the following conditions are equivalent for a separable Banach space X.

1. X is stable;
2. Every type on X is approximable;
3. Every type on X is strongly approximable.

By definition, every type on X is a pointwise limit of types realized in X. Thus, if X is separable, $\mathcal{T}(X)$ is separable with respect to the topology of pointwise convergence. It is a well-known fact that if X stable, then $\mathcal{T}(X)$ is *strongly separable*, *i.e.*, separable with respect to the topology of uniform convergence on bounded subsets of X. The converse was proved to be false by E. Odell (see [6, 8]). The preceding theorem explains to what extent stability of X is equivalent to approximability of types on X by types realized in X.

In Proposition 3.1, we characterize approximable functions in terms of finite representability: Let f be a real-valued function on X which is uniformly continuous on every bounded subset of X. Then the following conditions are equivalent.

1. f is approximable;
2. Whenever Y is finitely represented in X, there is a unique real-valued function g on Y such that (Y, g) is finitely represented in (X, f).

The proofs are based on ideas from model theory. Proposition 2.6 is inspired in the "Definability of Types" lemma in [7].

We will make heavy use of Banach space ultrapowers. For an introduction, we refer the reader to [3].

Throughout the paper, X denotes a normed space. If $M > 0$, we denote by $B(M)$ the set of elements of X of norm at most M.

2. Constructible Sets and Approximable Types

Let us first recall that a positive boolean combination of the sets $S_1, \ldots, S_n$ is a set obtained from $S_1, \ldots, S_n$ by taking finite unions and intersections.

2.1. Definition. Let X be a normed space. A *construction* C in X is a positive boolean combination of sets of the form

$$\{ x \in X \mid \|x + a_i\| \in I_i \}, \qquad a_1, \ldots, a_n \in X.$$

We write $C = C(a_1, \ldots, a_n; I_1, \ldots, I_n)$. If $I_1, \ldots, I_n = I$, we write $C = C(a_1, \ldots, a_n; I)$.

If $C(a_1,\dots,a_n;\, I_1,\dots,I_n)$ is a construction in X, we denote by

$$[\, C(a_1,\dots,a_n;\, I_1,\dots,I_n)\,] \tag{2}$$

the subset of X determined by C. We will call a subset X *constructible* if it is of the form (2). If $a_1,\dots,a_n$ are in a given subset A of X, we say that the set (2) is *constructible over* A.

Thus, the class of constructible subsets of X is the ring generated by the balls in X.

2.2. Definition. Let X be a normed space and let f be a real-valued function on X. We say that f is *approximable* if the following condition holds. For every choice of $M, \epsilon > 0$ and every interval I there exist a construction $C(a_1,\dots,a_n;\, J)$ and $\delta > 0$ such that

1. $B(M) \cap f^{-1}[I] \subseteq [\, C(a_1,\dots,a_n;\, J)\,]$;
2. $B(M) \cap [\, C(a_1,\dots,a_n;\, J + [-\delta,\delta])\,] \subseteq f^{-1}[I + [-\epsilon,\epsilon]]$.

If, regardless of the choice of M and ϵ, the set C can always be chosen constructible over a given subset A of X, we say that f is *approximable over* A.

We will express the fact that the inclusions (1) and (2) hold by saying that $[C(a_1,\dots,a_n;\, J)]$ *is* (ϵ,δ)*-equivalent to* $f^{-1}[I]$ *in the ball* $B(M)$.

Notice that if $f\colon X \to \mathbb{R}$ is approximable, then it is approximable over any given dense subset of X.

2.3. Proposition. *Let X be a normed space and let f be a real-valued function on X. The following conditions are equivalent.*

1. *f is approximable over A;*
2. *For every $M, \epsilon > 0$ and every interval I of the form $[\alpha,\infty)$ there exist a construction $C(a_1,\dots,a_n;\, J)$ with $a_1,\dots,a_n \in A$ and $\delta > 0$ such that $[C(a_1,\dots,a_n;\, J)]$ is (ϵ,δ)-equivalent to $f^{-1}[I]$ in $B(M)$;*
3. *For every $M, \epsilon > 0$ and every interval I of the form (α,∞) there exist a construction $C(a_1,\dots,a_n;\, J)$ with $a_1,\dots,a_n \in A$ and $\delta > 0$ such that $[C(a_1,\dots,a_n;\, J)]$ is (ϵ,δ)-equivalent to $f^{-1}[I]$ in $B(M)$;*
4. *For every $M, \epsilon > 0$ and every interval I of the form $(-\infty,\alpha]$ there exist a construction $C(a_1,\dots,a_n;\, J)$ with $a_1,\dots,a_n \in A$ and $\delta > 0$ such that $[C(a_1,\dots,a_n;\, J)]$ is (ϵ,δ)-equivalent to $f^{-1}[I]$ in $B(M)$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) is immediate, the equivalence (3) $\Leftrightarrow$ (4) follows by taking complements, and the implication (3)&(4) $\Rightarrow$ (1) is proved by taking intersections. □

Now we focus on a particular kind of real-valued functions, namely, types.

2.4. Definition. Let X be a normed space and let $\tau : X \to \mathbb{R}$ be a type on X. We will say that τ is *strongly approximable* if

· τ is approximable;
· The interval J of Definition 2.2 can always be taken arbitrarily close to I, and the norm of can be chosen $a_1,\dots,a_n$ arbitrarily close to the norm of τ.

2.5. Proposition. *Let X be a normed space and let τ be a type on X. The following conditions are equivalent.*

1. *τ is strongly approximable;*
2. *For every $M, \epsilon > 0$ and every interval of the form $[0,\alpha]$ there exist a construction $C(a_1,\dots,a_n;\, [0,\beta])$ and $\delta > 0$ such that*
   (i) $[\, C(a_1,\dots,a_n;\, [0,\beta])\,]$ is (ϵ,δ)-equivalent to $\tau^{-1}[0,\alpha]$ in $B(M)$;

(ii) $|\beta-\alpha|<\epsilon$ *and* $|\,\|a_i\|-\|\tau\|\,|<\epsilon$ *for* $i=1,\dots,n$.

Proof. Immediate from Definition 2.4 and (1) $\Leftrightarrow$ (4) of Proposition 2.3. □

2.6. Proposition. *Suppose that X is a stable Banach space. Then every type on X is strongly approximable.*

Proof. Let τ be a type on X. Take $M,\epsilon>0$ and an interval $[0,\alpha]$. We will define a construction $C(d_1,\dots,d_r;[0,\beta])$ and $\delta>0$ such that

(I) $B(M)\cap\tau^{-1}[0,\alpha]\subseteq[\,C(d_1,\dots,d_r;[0,\beta])\,]$;
(II) $B(M)\cap[\,C(d_1,\dots,d_r;[0,\beta+\delta])\,]\subseteq\tau^{-1}[0,\alpha+\epsilon]$.

Take β and δ such that

$$\alpha<\beta<\beta+\delta<\alpha+\epsilon$$

Without loss of generality, we can take δ such that

$$\delta<\min\{\,\beta-\alpha,(\alpha+\epsilon)-(\beta+\delta)\,\}. \tag{3}$$

Take also positive numbers $\eta,\eta_0,\eta_1,\dots$ such that

$$\delta<\eta_0<\eta_1<\dots<\eta$$

and η is less than the minimum in (3).

We will now construct, inductively,

· A sequence $a_0,a_1,\dots$ in $B(\tau(0)+\epsilon)$;
· For $i=-1,0,1,2,\dots$, sets $S(i)$, $T(i)$ of subsets of $\{0,\dots,i\}$;
· Elements $u^s_{i+1}\in B(M)$ for $s\in S(i)$ and $v^t_{i+1}\in B(M)$ for $t\in T(i)$.

Suppose that we have defined $a_0,a_1,\dots,a_n$, $S(-1),\dots,S(n-1)$, $T(-1),\dots,T(n-1)$, and u^s_i,v^t_i for $i=0,\dots,n$ and $s\in S(i)$, $t\in T(i)$. We now define the sets $S(n)$, $T(n)$ and the elements u^s_{i+1},v^t_{i+1}.

Let

$$S(n)=\Big\{\,s\subseteq\{0,\dots,n\}\;\Big|\;B(M)\cap\tau^{-1}[0,\alpha+\eta_n]\cap\bigcap_{i\in s}\tau_{a_i}^{-1}[\beta,\infty)\neq\emptyset\,\Big\}.$$

For each $s\in S(n)$, let u^s_{n+1} be an element of X such that

$$u^s_{n+1}\in B(M)\cap\tau^{-1}[0,\alpha+\eta_n]\cap\bigcap_{i\in s}\tau_{a_i}^{-1}[\beta,\infty).$$

Similarly, let

$$T(n)=\Big\{\,t\subseteq\{0,\dots,n\}\;\Big|\;B(M)\cap\tau^{-1}[\alpha+\epsilon-\eta_n,\infty)\cap\bigcap_{i\in t}\tau_{a_i}^{-1}[0,\beta+\delta]\neq\emptyset\,\Big\},$$

and for each $t\in T(n)$ let v^t_{n+1} be an element of X such that

$$v^t_{n+1}\in B(M)\cap\tau^{-1}[\beta+\epsilon-\eta_n,\infty)\cap\bigcap_{i\in t}\tau_{a_i}^{-1}[0,\beta+\delta].$$

We now define a_{n+1}. Let

$$F=\{\,u^s_{i+1}\mid -1\le i\le n,\ s\in S(i)\,\}\cup\{\,v^t_{i+1}\mid -1\le i\le n,\ t\in T(i)\,\}.$$

Since F is finite, there exists $a\in F\cap B(\tau(0)+\epsilon)$ such that

$$x\in F\cap\tau^{-1}[0,\alpha+\eta_n]\quad\text{implies}\quad\|a+x\|\in[0,\alpha+\eta_{n+1}]$$

and

$$x \in F \cap \tau^{-1}[\alpha + \epsilon - \eta_n, \infty)] \quad \text{implies} \quad \|a + x\| \in [\alpha + \epsilon - \eta_{n+1}, \infty).$$

Let a_{n+1} be such an element a.

2.7. *Claim.* Suppose that $0 \le i \le n$ and $s \in S(i-1)$, $t \in T(i-1)$. Then,

$$\|a_n + u_i^s\| \in [0, \alpha + \eta_n]$$

and

$$\|a_n + v_i^t\| \in [\alpha + \epsilon - \eta_n, \infty).$$

Claim 2.7 follows immediately from the preceding definitions.

2.8. *Claim.* Suppose that $0 \le i(0) < i(1) < \cdots < i(n)$ and

$$B(M) \cap \tau^{-1}[0, \alpha] \cap \bigcap_{j=0}^{n} \tau_{a_{i(j)}}^{-1}[\beta, \infty)] \neq \emptyset.$$

Then there exist $b_0, \ldots, b_n \in B(M)$ such that

$$\|a_{i(j)} + b_k\| \in [\beta, \infty), \quad \text{for } 0 \le j < k \le n$$

and

$$\|a_{i(j)} + b_k\| \in [0, \alpha + \eta], \quad \text{for } 0 \le k \le j \le n.$$

Proof of Claim 2.8. Inductively, we construct $b_0, \ldots, b_n$ such that

$$\|a_{i(j)} + b_k\| \in [\beta, \infty), \quad \text{for } 0 \le j < k \le n$$

and

$$\|a_{i(j)} + b_k\| \in [0, \alpha + \eta_{i(j)}], \quad \text{for } 0 \le k \le j \le n.$$

First we note that $S(i(0) - 1) \neq \emptyset$; in fact, $\emptyset \in S(i(0) - 1)$ since

$$B(M) \cap \tau^{-1}[0, \alpha + \eta_{i(0)}] \supseteq B(M) \cap \tau^{-1}[0, \alpha] \neq \emptyset.$$

Take $s \in S(i(0))$ and let b_0 be $u_{i(0)}^s$. Then, by Claim 2.7 above, we have

$$\|a_{i(j)} + b_0\| \in [0, \alpha + \eta_{i(j)}], \quad \text{for } 0 \le j \le n.$$

Assume that we have $b_1, \ldots, b_k$ as desired. Let $s = \{\, i(0), \ldots, i(k)\,\}$. From the definition of $S(i(k))$, we must have $s \in S(i(k))$. Let b_{k+1} be u_{k+1}^s. Then

$$\|a_{i(j)} + b_{k+1}\| \in [\beta, \infty)], \quad \text{for } 0 \le j \le k,$$

and by Claim 2.7,

$$\|a_{i(j+1)} + b_{k+1}\| \in [0, \alpha + \eta_{i(j+1)}], \quad \text{for } 0 \le k \le j \le n-1.$$

We have proved Claim 2.8.

2.9. *Claim.* Suppose that $0 \le i(0) < i(1) < \cdots < i(n)$ and

$$B(M) \cap \tau^{-1}[\alpha + \epsilon, \infty)] \cap \bigcap_{j=0}^{n} \tau_{a_{i(j)}}^{-1}[0, \beta + \delta] \neq \emptyset.$$

Then there exist $c_1, \ldots, c_n \in B(M)$ such that

$$\|a_{i(j)} + c_k\| \in [0, \beta + \delta], \quad \text{for } 0 \le j < k \le n$$

and

$$\|a_{i(j)} + c_k\| \in [\alpha + \epsilon - \eta, \infty), \quad \text{for } 0 \leq k \leq j \leq n.$$

Proof of Claim 2.9. The proof is analogous to that of Claim 2.8. We construct $c_1, \dots, c_n$ inductively such that

$$\|a_{i(j)} + c_k\| \in [0, \beta + \delta], \quad \text{for } 0 \leq j < k \leq n$$

and

$$\|a_{i(j)} + c_k\| \in [\alpha + \epsilon - \eta_{i(j)}, \infty), \quad \text{for } 0 \leq k \leq j \leq n.$$

2.10. *Claim.* There exists a number $N \in \mathbb{N}$ with the following property. Whenever $0 \leq i(0) < \cdots < i(N) \leq 2N$,

(i) There does not exist a sequence $(b_k)_{0 \leq k \leq N}$ in $B(M + \tau(0) + \epsilon)$ satisfying

$$\begin{aligned} &\|a_{i(j)} + b_k\| \in [\beta, \infty), \quad \text{for } 0 \leq j < k \leq N, \\ &\|a_{i(j)} + b_k\| \in [0, \alpha + \eta], \quad \text{for } 0 \leq k \leq j \leq N; \end{aligned} \tag{4}$$

(ii) There does exist a sequence $(c_k)_{0 \leq k \leq N}$ in $B(M + \tau(0) + \epsilon)$ satisfying

$$\begin{aligned} &\|a_{i(j)} + c_k\| \in [0, \beta + \delta], \quad \text{for } 0 \leq j \leq k \leq N, \\ &\|a_{i(j)} + c_k\| \in [\alpha + \epsilon - \eta, \infty), \quad \text{for } 0 \leq k < j \leq N. \end{aligned} \tag{5}$$

Proof of Claim 2.10. Suppose that the claim is false. Then, for arbitrarily large $N \in \mathbb{N}$ there will be $0 \leq i(0) < \cdots < i(N) \leq 2N$ and, either sequence $(b_k)_{0 \leq k \leq N}$ in $B(M + \tau(0) + \epsilon)$ such that (4) holds, or a sequence $(c_k)_{0 \leq k \leq N}$ in $B(M + \tau(0) + \epsilon)$ such that (5) holds. Now, for any given N there are finitely many choices for $0 \leq i(0) < \cdots < i(N) \leq 2N$. Hence, König's lemma provides a subsequence $(a_{n(l)})_{l \in \mathbb{N}}$ of (a_n) and, either a sequence $(b_k)_{k \in \mathbb{N}}$ in $B(M + \tau(0) + \epsilon)$ such that

$$\begin{aligned} &\|a_{i(l)} + b_k\| \in [\beta, \infty), \quad \text{for } 0 \leq l < k, \\ &\|a_{i(l)} + b_k\| \in [0, \alpha + \eta], \quad \text{for } 0 \leq k \leq l, \end{aligned}$$

or a sequence $(c_k)_{k \in \mathbb{N}}$ in $B(M + \tau(0) + \epsilon)$ such that

$$\begin{aligned} &\|a_{i(l)} + c_k\| \in [0, \beta + \delta], \quad \text{for } 0 \leq l \leq k, \\ &\|a_{i(l)} + c_k\| \in [\alpha + \epsilon - \eta, \infty), \quad \text{for } 0 \leq k < l. \end{aligned}$$

Either case contradicts the stability of X. Claim 2.10 is proved.

Fix N as in Claim 2.10. Define

$$\{ d_1, \dots, d_r \} = \{ a_{i(j)} \mid 0 \leq i(0) < \cdots < i(N) \leq 2N, \ 0 \leq j \leq N \}$$

and

$$C(d_1, \dots, d_r; [0, \beta]) = \bigcup_{0 \leq i(0) < \cdots < i(N) \leq 2N} \ \bigcap_{0 \leq j \leq N} \tau_{a_{i(j)}}[0, \beta]. \tag{6}$$

Condition (II) follows directly from Claim 2.9 and the choice of N. To prove (I), suppose that $x \in B(M)$ and $x \notin [C]$. Fix one of the intersections in (6). The element x is not in this intersection, so there exists an index $i(j_0)$ such that $x \notin \tau_{a_{i(j_0)}}[0, \beta]$. Now take an N-element subset of $\{1, \dots, 2N\}$ not containing $a_{i(j_0)}$ and consider the intersection corresponding to this set in (6). Repeat the argument to find $i(j_1)$ distinct from $i(j_0)$ such that $x \notin \tau_{a_{i(j_1)}}[0, \beta]$. The argument can be iterated NÝ times. But then, Claim 2.8 and the choice of N imply $x \notin \tau^{-1}[0, \alpha]$. ◻

Remark. It is well-known that the space of types of a stable Banach space is *strongly separable*, *i.e.*, separable with respect to the topology of uniform convergence on bounded sets. (The converse is not true; see [6, 8].) This is immediate from Proposition 2.6. In fact, it is easy to see that if every type on X is approximable, then the density of $\mathcal{T}(X)$ with respect to the strong topology must equal the density of X (with respect to the norm topology).

3. Approximable Functions

Let X be a normed space and let f be a real-valued function on X which is uniformly continuous on every bounded subset of X. An *ultrapower* of (X, f) is defined as follows. If $\mathcal{U}$ is an ultrafilter, $(\hat{X}, \hat{f})$ is the ultrapower of (X, f) with respect to $\mathcal{U}$ if

- $\hat{X}$ is the ultrapower of X with respect to $\mathcal{U}$;
- Whenever $x \in \hat{X}$ and $(x_i)_{i\in I}$ is representative of x in $\hat{X}$, we have $\hat{f}(x) = \lim_{\mathcal{U}}(f(x_i))_{i\in I}$.

The fact that f is uniformly continuous on the bounded subsets of X ensures that $\hat{f}$ is well-defined.

An ultrapower $(\hat{X}, \hat{f})$ of (X, f) has the property that it is *finitely represented in* (X, f). This means that whenever E is a finite dimensional subspace of $\hat{X}$ and $M, \epsilon > 0$, there exists a finite dimensional subspace F of X such that $(E, \hat{f} \restriction E)$ and $(F, f \restriction E)$ are $(1+\epsilon)$-*isomorphic* in the sense that there exists a $(1+\epsilon)$-isomorphism $\varphi\colon E \to F$ satisfying $|f(\varphi(x)) - \hat{f}(x)| \le \epsilon$ for every $x \in E$ of norm at most M.

Let X and Y be normed spaces containing a common subset A. If $\epsilon > 0$, we say that X and Y are $(1+\epsilon)$-*isomorphic over* A if there exists a $(1+\epsilon)$-isomorphism $\varphi\colon X \to Y$ such that such that $\varphi \restriction A$ is the identity. We will say that Y is A-finitely represented in X if the following condition holds. Given $\epsilon > 0$ and a finite dimensional subspace F of Y, there exists a subspace E of X such that the spaces $\overline{\text{span}}[E \cup A]$ and $\overline{\text{span}}[F \cup A]$ are $(1+\epsilon)$-isomorphic over A.

We will now characterize approximability of real-valued functions in terms of finite representability. Let us first notice the following.

Remarks.

1. If X and Y contain a common subset A and Y is A-finitely represented in X, then there is an ultrapower $(\hat{X}, \hat{f})$ of (X, f) and an embedding $\varphi\colon Y \to \hat{X}$ which fixes A pointwise.
2. If $(\hat{X}, \hat{f})$ is an ultrapower of (X, f) and f is approximable over A, then so is $\hat{f}$; in fact, if $0 < M < M'$, $0 < \epsilon < \epsilon' < \epsilon''$, and $0 < \delta < \delta' < \delta''$ are such that $\big[\, C(a_1, \dots, a_n;\, J)\,\big]_X$ is $(\epsilon' - \epsilon, \delta'')$-equivalent to $f^{-1}[I + [-\epsilon, \epsilon]]$ in the ball $B_X(M')$, then $\big[\, C(a_1, \dots, a_n;\, J + [-\delta, \delta]\,\big]_{\hat{X}}$ is $(\epsilon'', \delta' - \delta)$-equivalent to $\hat{f}^{-1}[I]$ in the ball $B_{\hat{X}}(M)$.

3.1. Proposition. *Let X be a normed space and let f be a real-valued function on X which is uniformly continuous on every bounded subset of X. Then, if A is a subset of X, the following conditions are equivalent.*

1. *f is approximable over A;*
2. *Whenever $Y \supseteq A$ and Y is A-finitely represented in X, there is a unique real-valued function g on Y such that (Y, g) is A-finitely represented in (X, f).*

Proof. (1) $\Rightarrow$ (2) follows easily from the preceding remarks. We prove (2) $\Rightarrow$ (1).

Suppose that f is not approximable over A. Take $M, \epsilon > 0$ and an interval I such that there do not exist $[\, C(a_1, \dots, a_n;\, J)\,]$ with $a_1, \dots, a_n \in A$ and $\delta > 0$ with $[\, C(a_1, \dots, a_n;\, J)\,]$

(ϵ, δ)-equivalent to $f^{-1}[I]$ in the ball $B(M)$. Without loss of generality, we can assume that I is bounded.

Let

$$\mathfrak{C} = \big\{ C(a_1, \dots, a_n; J) \ \mid\ a_1, \dots, a_n \in A \text{ and } B(M) \cap f^{-1}[I] \subseteq [\, C(a_1, \dots, a_n; J)\,] \big\}.$$

By our assumption, whenever $C(a_1, \dots, a_n; J) \in \mathfrak{C}$ and $\delta > 0$,

$$B(M) \cap \Big([\, C(a_1, \dots, a_n; J + [-\delta, \delta])\,] \cap \complement f^{-1}[I + [-\epsilon, \epsilon]] \Big) \neq \emptyset.$$

Also, $\mathfrak{C}$ is closed under finite intersections. Hence, there exists an ultrapower $(\hat{X}, \hat{f})$ of (X, f) and $b \in \hat{X}$ such that

$$b \in B(M) \cap \bigcap_{C(a_1,\dots,a_n; J) \in \mathfrak{C}} [\, C(a_1, \dots, a_n; J)\,] \cap \complement \hat{f}^{-1}[I + [-\epsilon/2, \epsilon/2]].$$

Now, notice that if $a_1, \dots, a_n \in A$ and $b \in [\, C(a_1, \dots, a_n; (-\infty, \alpha])\,]$, for every $\beta > \alpha$ we must have

$$B(M) \cap f^{-1}(I) \cap [\, C(a_1, \dots, a_n; (-\infty, \beta])\,] \neq \emptyset$$

(otherwise, $[\, C(a_1, \dots, a_n; [\beta, \infty))\,] \in \mathfrak{C}$ and $b \in [\, C(a_1, \dots, a_n; [\beta, \infty))\,]$, which is impossible). Hence, there exists an ultrapower $(\hat{X}', \hat{f}')$ of (X, f) and $b' \in \hat{X}'$ such that

(i) $\hat{f}'(b') \in I$;
(ii) $b' \in [\, C(a_1, \dots, a_n; (-\infty, \alpha])\,]$ whenever $a_1, \dots, a_n \in A$ and $b \in [\, C(a_1, \dots, a_n; (-\infty, \alpha])\,]$.

By (ii), there is an isometry between $\overline{\text{span}}[\{b\} \cup A]$ and $\overline{\text{span}}[\{b'\} \cup A]$ mapping b to b' and fixing A pointwise. But $\overline{\text{span}}[\{b\} \cup A]$ and $\overline{\text{span}}[\{b'\} \cup A]$ are A-finitely represented in X and $\hat{f}(b) \notin I$, so we are in contradiction with (2). □

4. Approximable Types and Stability

We now prove the main result.

4.1. Theorem. *Let X be a separable Banach space. Then the following conditions are equivalent.*

1. *X is stable;*
2. *Every type on X is approximable;*
3. *Every type on X is strongly approximable.*

Proof. (1) $\Rightarrow$ (3) is Proposition 2.6. We prove (2) $\Rightarrow$ (1).

Suppose that X is not stable. Then there exist bounded sequences (a_m) and (b_n) in X and real numbers α, β such that

$$\sup_{m<n} \|a_m + b_n\| \leq \alpha < \beta \leq \inf_{n<m} \|a_m + b_n\|. \tag{7}$$

Without loss of generality, we can assume that (a_m) is *type determining*, *i.e.*, there exists a type $\tau \in \mathcal{T}(X)$ such that $\tau(x) = \lim_{m\to\infty} \|a_m + x\|$ for every $x \in X$.

By (7) there exists an ultrapower $\hat{X}$ of X, an element $a \in \hat{X}$, and types ρ_1, ρ_2 on $\hat{X}$ such that

- $(\hat{X}, \rho_1)$ and $(\hat{X}, \rho_2)$ are finitely represented in (X, τ);
- $\rho_1(a) \leq \alpha$ and $\rho_2(a) \leq \beta$.

But then τ cannot be approximable, by Proposition 3.1. □

Remark. The concepts considered here are particularizations of concepts from the logical analysis of stability in [4]. Indeed, the notions of type, constructible subset, and approximable function correspond (respectively) to the "quantifier-free" versions of the notions of *type*, *definable subset*, and *definable real-valued relation* considered in [4].

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