STABLE BANACH SPACES AND BANACH SPACE STRUCTURES, II: FORKING AND COMPACT TOPOLOGIES

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ABSTRACT. We study model theoretical stability for structures from functional analysis. We prove a functional-analytic version of the Finite Equivalence Relation Theorem. We also the Stability Spectrum Theorem for Banach space structures.

1. INTRODUCTION

In [3], we introduced the notion of model theoretical stability for Banach space structures and gave several characterizations of stability. Here we study further properties of stable theories.

In Sections 3 and 4 we introduce the concept forking for positive bounded formulas and prove some properties which characterize forking. The proofs in these sections are direct adaptations of the corresponding proofs in classical model theory, but we include them for completeness.

In Section 5 we prove a functional analytic version of the Finite Equivalence Relation Theorem:

Theorem. *Let $p \in S_n(A)$ and q_1 and q_2 be two distinct nonforking extensions of p over a model E containing A. Then, if $N > \|p\|$, there exists a pseudometric ρ on $(\mathcal{B}_N)^n$ such that*

1. *ρ is definable over A;*
2. *$(\mathcal{B}_N)^n$ is compact with respect to ρ;*
3. *There exists $\epsilon > 0$ such that $\rho(\bar{a}_1, \bar{a}_2) > \epsilon$, whenever $\bar{a}_1$ realizes q_1, and $\bar{a}_2$ realizes q_2.*

We call this result the "Compact Pseudometric Theorem". The role played by equivalence relations in classical model theory is mirrored in this context by pseudometrics, and the role played by finiteness is mirrored by compactness.

Section 7 is devoted to superstability. To characterize superstability in terms of forking in this context, a topological analysis of forking is needed. Namely, if an extension forks, "how much" does it fork? This type of analysis is developed in Section 6.

In Section 8 we prove that the Stability Spectrum Theorem for Banach Space structures for the case when the uniform structure is metrizable. Finally, in Section 9, we provide two examples: an example of a Banach space structure which is superstable but not ω-stable with respect to the metric d on the space types, and an example of a structure which is stable but not superstable with respect to the same metric. Both structures consist of Hilbert spaces with operators.

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We extend to this paper the assumptions and notational conventions of [3]. We deal with Banach space structures, positive bounded formulas, and approximate satisfaction $\models_{\mathcal{A}}$. If φ and ψ are positive bounded formulas, $\varphi < \psi$ means that ψ is an approximation of φ. If Σ is a set of positive bounded formulas, Σ_+ denotes the set of approximations of formulas in Σ.

Throughout the paper, T denotes a *stable* positive bounded theory over a countable language. The letter $\mathfrak{C}$ denotes a monster model for T. The norm of a finite tuple in $\mathfrak{C}$ is its ℓ_∞-norm. The norm of a type is the norm of a tuple realizing the type. If X is a subset of $\mathfrak{C}$ or a set of types, $\mathcal{B}_N(X)$ denotes the set of elements of X of norm less than or equal to N. We write $\mathcal{B}(X)$ instead of $\mathcal{B}_1(X)$.

The letter $\mathfrak{U}$ denotes a fixed uniform structure on the space of types of T. Unless specified otherwise, all the vicinities mentioned are vicinities of $\mathfrak{U}$.

2. The topology of formulas

If φ is an $L(\mathfrak{C})$-formula, we let $[\,\varphi\,] = \{\, p \in S(T) \mid \varphi \in p\,\}$. The *logical topology* on $S(T)$ is defined as follows. The basic closed sets are the sets of the form $[\,\varphi\,]$. This topology is Hausdorff. The sets $[\,\varphi\,]$ need not be open. However, if $p \in S(T)$, the sets $\{\,[\,\psi\,] \mid \psi \in p_+\,\}$ form a basic system of neighborhoods of p.

The logical topology is not compact: For $m < \omega$, let $\varphi_m(x)$ be the formula $\|x\| \geq m$. Then $[\,\varphi_0\,] \supseteq [\,\varphi_1\,] \supseteq \dots$, but $\bigcap_{m<\omega} [\,\varphi_m\,] = \emptyset$. However, by the compactness theorem, the restriction of the logical topology to $\mathcal{B}_N(\,S(T)\,)$ is compact, for every $N > 0$. Hence, the logical topology is locally compact and σ-compact.

If $A \subseteq B$, the restriction map from $S(B)$ onto $S(A)$ is continuous with respect to the logical topology. Since $\mathcal{B}_N(\,S(B)\,)$ is compact, for every $N > 0$, the restriction map from $\mathcal{B}_N(\,S(B)\,)$ onto $\mathcal{B}_N(\,S(B)\,)$ is closed.

In this paper, when we refer to topological properties of sets of types, the underlying topology is assumed to be the logical topology.

3. Forking

In this section and the next, we define the concept forking in Banach space model theory and prove some properties which characterize it. Any of the approaches to the calculus of forking available in the literature can be transposed, more or less straightforwardly, into the context Banach space model theory. Nonetheless, we have included the proofs for reference, inasmuch as they are short. We have followed the approach of M. Ziegler's lecture notes [8].

Let $(\,x_i \mid i \in I\,)$ be a sequence of variables. A a *type in* $(\,x_i \mid i \in I\,)$ is a set p of formulas in the variables $(\,x_i \mid i \in I\,)$ such that for every $n \geq 0$ and $i_1, \dots, i_n \in I$, the restriction of p to formulas in the variables $x_{i_1}, \dots, x_{i_n}$ is a type.

If $A \subseteq \mathfrak{C}$, the set of types over A in the variables $(\,x_i \mid i \in I\,)$ is denoted $S_I(A)$.

We shall use boldface letters $\mathbf{p}, \mathbf{q}, \dots$ to denote types in $S_I(\mathfrak{C})$. The barred letters $\bar{a}, \bar{b}, \dots$ will denote sequences indexed by I, regardless of the cardinality of I.

Let $E \subseteq A$ and let $\mathbf{p} \in S_I(A)$. We say that p is *finitely realized in E* if for every positive bounded formula $\varphi(\bar{x}, \bar{a}) \in p$ and every $\psi > \varphi$ there exists $\bar{u} \in E$ such that $E \models_{\mathcal{A}} \psi(\bar{u}, \bar{a})$. (This concept was introduced in [3] for types in a finite number of variables.)

3.1. Definition. Let $A \subseteq \mathfrak{C}$, and $\mathbf{p} \in S_I(\mathfrak{C})$. We say that $\mathbf{p}$ *does not fork over A* if $\mathbf{p}$ is finitely realized in every model containing A.

Proposition 3.2. *Let $A \subseteq \mathfrak{C}$. Every type over A has an extension over $\mathfrak{C}$ which does not fork over A.*

Given a sequence of variables $(\, x_i \mid i \in I\,)$, we will denote by $\mathcal{N}_A(\, x_i \mid i \in I\,)$ the following set:

$$\left\{ \sigma(x_{i_1},\dots,x_{i_n},\bar{b}) \;\middle|\; \begin{array}{c} i_1,\dots,i_n \in I, \bar{b} \in \mathfrak{C} \\ \text{There exists a model } E \supseteq A \text{ such that } E^n \subseteq \sigma(\mathfrak{C},\bar{b}) \end{array} \right\}.$$

3.3. Lemma (Fundamental Existence Lemma)**.** *If* $\mathbf{p} \in S_I(\mathfrak{C})$ *and*

$$\mathbf{p} \supseteq \mathcal{N}_A(\, x_i \mid i \in I\,),$$

then $\mathbf{p}$ *does not fork over* A.

Proof. Let E be a model containing A. We wish to show that if $\varphi(\bar{x},\bar{b}) \in \mathbf{p}$ and $\psi > \varphi$, then there exists $\bar{u} \in E$ such that $E \models_{\mathcal{A}} \psi(\bar{u},\bar{b})$. Take such φ and ψ. If there is no $\bar{u} \in E$ as above, then $E \subseteq \operatorname{neg}(\psi(\mathfrak{C},\bar{b}))$, so $\operatorname{neg}(\psi(\bar{x},\bar{b})) \in \mathcal{N}_A(\, x_i \mid i \in I\,) \subseteq \mathbf{p}$. But this contradicts the consistency of $\mathbf{p}$. ⊣

Proof of Proposition 3.2. By the Fundamental Existence Lemma, we only have to show that $p \cup \mathcal{N}_A$ is consistent. Suppose, by way of a contradiction, that it is inconsistent.

Claim. *There exist* $N > 0$*, an* $L(A)$*-formula* $\psi(\bar{x})$*, models* $E_1,\dots,E_m$ *containing* A*, and* $L(\mathfrak{C})$*-formulas* $\sigma_1(\bar{x},\bar{b}_1),\dots,\sigma_m(\bar{x},\bar{b}_m)$ *(with all the parameters exhibited) such that*

- $\|\bar{x}\| \le N \wedge \psi(\bar{x})$ *is realized in every model;*
- $E_i \subseteq \sigma_i(\mathfrak{C},\bar{b}_i)$*, for* $i = 1,\dots,m$*;*
- $\psi(\bar{x}) \wedge \bigwedge_{i=1}^m \sigma_i(\bar{x},\bar{b}_i)$ *is inconsistent.*

Proof of the Claim. Suppose that $p \cup \mathcal{N}_A(\, x_i \mid i \in I\,)$ is inconsistent. Then there exist $\varphi \in p_+$ and $\sigma_1(\bar{x},\bar{c}_1),\dots,\sigma_m(\bar{x},\bar{b}_m) \in \mathcal{N}_A$, such that

$$\varphi(\bar{x}) \wedge \bigwedge_{i=1}^{m} \sigma_i(\bar{x},\bar{b}_i)$$

is inconsistent. We can assume that φ is of the form $\|\bar{x}\| \le N \wedge \psi(\bar{x})$, where N is larger than the norm of the restriction of $\mathbf{p}$ to the variables $\bar{x}$. ⊣

Now we show that the claim contradicts the stability of T. Fix an infinite cardinal κ, in order to prove that T is κ-unstable with respect to the discrete uniform structure. Let λ be the least cardinal such that $2^\lambda > \kappa$.

A simple compactness argument shows that can assume that the E_i's of the claim are isomorphic to $\mathfrak{C}$. This assumption allows us to apply the claim iteratively. We find models E_s, for $s \in m^{<\lambda}$, and tuples $\bar{b}_{s\frown 1},\dots,\bar{b}_{s\frown m} \in E_s$, such that

(i) $E_t \subseteq E_s$, if $s \subset t$;
(ii) $\|\bar{x}\| \le N \wedge \psi(\bar{x})$ is realized in each E_s;
(iii) $E_{s\frown i} \subseteq \wedge\sigma_i(E_s,\bar{b}_{s\frown i})$, for $s \in m^{<\lambda}$ and $i = 1,\dots,m$;
(iv) $\psi(\bar{x}) \wedge \bigwedge_{i=1}^m \sigma_i(\bar{x},\bar{b}_s)$ is inconsistent, for every $s \in m^{<\lambda}$.

Let $A = \bigcup\{\, \bar{b}_s \mid s \in m^{<\lambda}\,\}$. We have $\operatorname{card}(A) \le \kappa$. For each $i = 1,\dots,m$ and each $\xi \in m^\lambda$, the set

$$\{\, \|\bar{x}\| \le N \wedge \psi(\bar{x})\,\} \cup \{\, \sigma_i(\bar{x},\bar{b}_{s\frown i}) \mid s \frown i \subset \xi\,\}$$

is consistent, by (i)–(iii). Let q_ξ be a type extending this set. By (iv), $q_\xi \neq q_\eta$ if ξ and η are distinct sequences in m^λ. Hence, $\operatorname{card}(S_n(A)) \ge 2^\lambda > \kappa$, so T is not κ-stable with respect to the discrete uniform structure. ⊣

Let $p(\bar{x})$ and $q(\bar{x})$ be types over $\mathfrak{C}$. Let

$$(\mathfrak{C}, \mathcal{P}_\varphi)_{\varphi(\bar{x})\in L}, \qquad (\mathfrak{C}, \mathcal{Q}_\varphi)_{\varphi(\bar{x})\in L}$$

be corresponding morleyizations (see[3]). . For $N > 0$, we define

$$\mathrm{dist}(\ p|\mathcal{B}_N,\ q|\mathcal{B}_N\ ) = \sup\left\{ \|\, \mathcal{P}_\varphi|\mathcal{B}_N \ -\ \mathcal{Q}_\varphi|\mathcal{B}_N \,\|_\infty \ \middle|\ \varphi(\bar{x}) \in L \right\}.$$

Then dist is a metric on $S(\mathcal{B}_N)$.

The following concept was introduced in [4].

3.4. Definition. A type p over $\mathfrak{C}$ is *almost over* A if $\mathrm{conj}^A(p)$ is a compact subset of $(\ S(\mathcal{B}_N),\ \mathrm{dist}\ )$, for every $N > 0$.

Proposition 3.5. *Let A be a subset of $\mathfrak{C}$, and let $\mathbf{p} \in S_I(\mathfrak{C})$. The following conditions are equivalent.*

(1) $\mathbf{p}$ does not fork over A.
(2) $\mathbf{p}$ is the heir of $\mathbf{p}|E$, for every model E containing A.
(3) $\mathbf{p}$ is definable over every model containing A.
(4) $\mathbf{p}$ is almost definable over A.
(5) $\mathbf{p}$ has at most $2^{\aleph_0}$-many A-conjugates.
(6) There exists a cardinal κ such that $\mathbf{p}$ has at most κ many A-conjugates.

Proof. Since T is stable, $\mathbf{p}$ is definable. Thus, (3)–(5) are equivalent by Corollary 8 of [4]. The implication (2) $\Rightarrow$ (1) is given by Theorem 8.7 of [3], and (3) $\Rightarrow$ (2) is clear. We now prove (1) $\Rightarrow$ (3).

Let E be a model containing A. We show that every E-automorphism of $\mathfrak{C}$ fixes $\mathbf{p}$. The Beth Definability Theorem [2] will then imply that $\mathbf{p}$ is definable over E. Let f be such an automorphism. Take a formula $\varphi(\bar{x}, \bar{b}) \in \mathbf{p}$, a formula $\psi(\bar{x},\ f(\bar{c})) \in f(\mathbf{p})$, and approximations $\varphi' > \varphi$ and $\psi' > \psi$. Since $\mathbf{p}$ is finitely realized in E, there exists $\bar{u} \in E$ such that $\models \varphi'(\bar{u}, \bar{b}) \wedge \psi'(\bar{u}, \bar{c})$. Since f fixes E pointwise, we also have $\models \varphi'(\bar{u}, \bar{b}) \wedge \psi'(\bar{u},\ f(\bar{c}))$. We have shown that $\mathbf{p}_+ \cup (\ f(\mathbf{p}))_+$ is consistent. We conclude, then, that $f(\mathbf{p}) = \mathbf{p}$. ⊣

4. Properties of forking

4.1. Definition. Let $A \subseteq B$, and $p \in S(B)$. We say that *p does not fork over A*, or that *p is a nonforking extension of $p|A$*, if p has an extension $\mathbf{p}$ over $\mathfrak{C}$ such that $\mathbf{p}$ does not fork over A.

Property 1 (Uniqueness)**.** *Let $A \subseteq B$ and $p, q \in S(B)$. If p and q do not fork over A and $p|A = q|A$, then p and q are conjugates over A.*

First we show the following lemma.

4.2. Lemma. *Suppose that $\mathrm{tp}(\bar{a}/A \cup \bar{b})$ does not fork over A. Then every extension of $\mathrm{tp}(\bar{b}/A)$ over $\mathfrak{C}$ is an A-conjugate of some extension of $\mathrm{tp}(\bar{b}/A \cup \bar{a})$.*

Proof of the Lemma. The hypothesis means that there exists an extension $\mathbf{p}$ of $\mathrm{tp}(\bar{a}/A \cup \bar{b})$ which does not fork over A.

Let $\mathbf{q}$ be an extension of $\mathrm{tp}(\bar{b}/A)$. We seek a conjugate $\mathbf{q}'$ of $\mathbf{q}$, extending $\mathrm{tp}(\bar{b}/A \cup \bar{a})$.

Claim. *Let E be a model containing A. Then*

(1) There exists an A-automorphism f of $\mathfrak{C}$ such that $f(\mathbf{q})$ extends $\mathrm{tp}(\ \bar{b}/f(E)\ )$.
(2) There exists an $A \cup \bar{b}$-automorphism g of $\mathfrak{C}$ such that $g(\mathbf{p})$ extends $\mathrm{tp}(\bar{a}/g(\ f(E)) \cup \bar{b})$.

Proof of the claim. (1): Take $\bar{b}_0$ such that $\mathbf{q}$ extends $\mathrm{tp}(\bar{b}_0/E)$. Let f be an A-automorphism of $\mathfrak{C}$ mapping $\bar{b}_0$ to $\bar{b}$. Then $f(\mathbf{q})$ extends $\mathrm{tp}(\,\bar{b}/f(E)\,)$.

The proof of (2) is similar. ⊣

Take now a model $E \supseteq A$ such that $\mathbf{q}$ is definable over E. Let f and g correspond to E as in the claim, and let $\mathbf{q}' = g(f(\mathbf{q}))$. Then $\mathbf{q}'$ is definable over $g(f(E))$, so $\mathbf{q}'$ is the heir of $\mathbf{q}'|g(f(E)) = \mathrm{tp}(\,\bar{b}\,/\,g(f(E))\,)$. Also, by hypothesis $\mathrm{tp}(\,\bar{a}\,/\,g(fE)) \cup \bar{b}\,)$ is finitely realized in $g(f(E))$. This means that $\mathrm{tp}(\,\bar{b}\,/\,(g(f(E)) \cup \bar{a}\,)$ is the heir of $\mathrm{tp}(\,\bar{b}\,/\,g(f(E))\,)$. But heirs are unique, so $\mathbf{q}'$ must extend $\mathrm{tp}(\,\bar{b}\,/\,g(f(E))\,)$. ⊣

Proof of Property 1. Let $p = \mathrm{tp}(\bar{b}/A)$. By Proposition 3.2, there exists a model $E \supseteq A$ such that $\mathrm{tp}(E/A \cup \bar{b})$ does not fork over A. Let $\mathbf{p}$ be the heir of $\mathrm{tp}(\bar{b}/E)$. We show that every extension $\mathbf{q}$ of p which does not fork over A is an A-conjugate of $\mathbf{p}$. Take such an extension $\mathbf{q}$. By the preceding lemma, there is an A-conjugate $\mathbf{q}'$ of $\mathbf{q}$ that extends $\mathrm{tp}(\bar{b}/E)$. By Proposition 3.5, $\mathbf{q}'$ is the heir of $\mathrm{tp}(\bar{b}/E)$. Thus $\mathbf{q}' = \mathbf{p}$. ⊣

Property 2 (Isomorphism)**.** *Let $A \subseteq B$ and $p \in S(B)$. If p does not fork over A and q is A-isomorphic to p, then q does not fork over A.*

Proof. Clear. ⊣

Property 3 (Heir)**.** *A type p does not fork over a model E if and only if p is the heir of $p|E$.*

Proof. By Proposition 3.5. ⊣

Property 4 (Existence)**.** *Let $A \subseteq B$. Every type over A has an extension over B which does not fork over A.*

Proof. By Proposition 3.2. ⊣

Property 5 (Monotonicity)**.** *Let $A \subseteq B \subseteq C$ and $p \in S(C)$. If p does not fork over A, then p does not fork over B and $p|B$ does not fork over A.*

Proof. Clear. ⊣

Property 6 (Continuity)**.** *Let $p \in S(B)$.*

(1) If $A \subseteq B$, $p(\bar{x}) \in S(B)$ and p forks over A, there exists a formula $\varphi \in p_+$ such that every extension of $p|A$ containing φ forks over A.

(2) For every $p(\bar{x}) \in S(B)$ there exists a countable subset of B over which p does not fork.

Proof. (1): An extension $q(\bar{x})$ of $p|A$ is nonforking if and only if q is consistent with $p|A \cup \mathcal{N}_A(\bar{x})$. Now, p is a forking extension of $p|A$; hence, there exists a formula $\varphi \in p_+$ such that $\{\varphi\}$ is inconsistent with $p|A \cup \mathcal{N}_A(\bar{x})$. Any extension of $p|A$ containing φ must fork over A.

(2): Let $\mathbf{p}(\bar{x})$ be an extension of p which does not fork over B. By Proposition 3.5, $\mathbf{p}$ is almost definable over B. But then $\mathbf{p}$ is almost definable over a countable subset B_0 of B.(See [4].) Again by Proposition 3.5, we conclude that p does not fork over B_0. ⊣

Property 7 (Boundedness)**.** *If $A \subseteq B$, every type over A has at most $2^{\aleph_0}$ nonforking extensions over B.*

Proof. From Proposition 3.5 and Uniqueness. ⊣

Property 8 (Transitivity)**.** *Let $A \subseteq B \subseteq C$ and $p \in S(C)$. If p does not fork over B and $p|B$ does not fork over A, then p does not fork over A.*

Proof. Take an extension $\mathbf{p}$ of p and an extension $\mathbf{q}$ of $p|B$ such that $\mathbf{p}$ does not fork over B and $\mathbf{q}$ does not fork over A. Then $\mathbf{p}$ does not fork over B, so, by Uniqueness, $\mathbf{p}$ and $\mathbf{q}$ are B-conjugate. Thus, $\mathbf{p}$ does not fork over A. ⊣

4.3. Definition. Let A, B, C be subsets of $\mathfrak{E}$. We say that A and B are *independent over* C if $\text{tp}(A/C \cup B)$ does not fork over C.

Proposition 4.4. *(1)* (Symmetry) *A and B are independent over C if and only if B and A are independent over C.*
(2) $A_1 \cup A_2$ and B are independent over C if and only if A_1, B are independent over C, and A_2, B are independent over C.

Proof. (1): Suppose that A and B are independent over C. Take an extension $\mathbf{p}$ of $\text{tp}(A/C \cup B)$ which does not fork over C. By Lemma 4.2 there exists an extension $\mathbf{q}$ of $\text{tp}(B/C \cup A)$ such that $\mathbf{q}$ is a C-conjugate of $\mathbf{p}$. Hence, $\text{tp}(B/C \cup A)$ does not fork over C.

(2): The following conditions are equivalent (the second equivalence is a consequence of Monotonicity and Transitivity):

- $A_1 \cup A_2$ and B are independent over C;
- A_1, B are independent over $A_2 \cup C$, and A_2, B are independent over C;
- A_1, B are independent over C, and A_2, B are independent over C.

⊣

For $A \subseteq B$ and $M > 0$, let

$$\mathcal{N}_M(B, A) = \{ p \in \mathcal{B}_M(S(B)) \mid p \text{ does not fork over } A \}.$$

Proposition 4.5 (Open Map Theorem)**.** *If $A \subseteq B$ and $M > 0$, the restriction map from $\mathcal{N}_M(B, A)$ onto $\mathcal{B}_M(S(A))$ is open.*

Proof. Without loss of generality, we can assume $B = \mathfrak{E}$. Let U be an open subset of $\mathcal{N}_M(\mathfrak{E}, A)$. Let

$$\tilde{U} = \{ \mathbf{q} \in \mathcal{N}_M(\mathfrak{E}, A) \mid \mathbf{q}|A \in U|A \}.$$

By Uniqueness, $\tilde{U}$ is a union of Aconjugates of U, and hence it is open. Therefore, $\mathcal{B}_M(S(A)) \setminus U|A = (\mathcal{N}_M(\mathfrak{E}, A) \setminus \tilde{U})|A$ is closed, and $U|A$ is open. ⊣

5. The Compact Pseudometric Theorem

One of the goals of this paper is to point out the analogy between the role played by pseudometrics in the model theory of Banach space structures and the role played by equivalence relations in classical model theory; also, the analogy between the role played by *compact* pseudometrics in analysis and that played by *finite* equivalence relations in algebra. One of the central results of classical stability theory is Shelah's Finite Equivalence Relation Theorem [6]. In this section we prove the analogous result, involving compact pseudometrics, for Banach space structures:

5.1. Theorem (Compact Pseudometric Theorem)**.** *Let $p \in S_n(A)$ and q_1 and q_2 be two distinct nonforking extensions of p over a model E containing A. Then, if $N > \|p\|$, there exists a pseudometric ρ on $(\mathcal{B}_N)^n$ such that*

(1) ρ is definable over A;
(2) $(\mathcal{B}_N)^n$ is compact with respect to ρ;

(3) There exists $\epsilon > 0$ such that $\rho(\bar{a}_1, \bar{a}_2) > \epsilon$, whenever $\bar{a}_1$ realizes q_1, and $\bar{a}_2$ realizes q_2.

In the proof we will use the fact (proved by A. Weil in his classic monograph [7]) that a uniform structure that has a countable base is metrizable. Furthermore, we will use some fine aspects the construction given in the proof.

Proposition 5.2. *Let $\mathfrak{V}$ be a uniform structure on X. Let $\{ U_m \mid m < \omega \}$ be a sequence of subsets of $X \times X$ such that:*

- *$U_0 = X \times X$;*
- *U_m is a vicinity of $\mathfrak{V}$, for each $m > 0$;*
- *$U_{m+1} \circ U_{m+1} \circ U_{m+1} \subseteq U_m$ for every m.*

Then there exists a pseudometric ρ on X such that

(1) $U_m \subseteq \{ (x, y) \mid \rho(x, y) \leq 2^{-m} \} \subseteq U_{m-1}$, for each $m > 0$;
(2) If f is a bijection of X such that $f(U_m) = U_m$ for every m, then $f(\rho) = \rho$.

If every vicinity of $\mathfrak{V}$ contains some U_m, then it follows from (1) that $\mathfrak{V}$ is the uniform structure of ρ. In particular, if the topology of $\mathfrak{V}$ is Hausdorff, then ρ is a metric.

For the proof of the preceding proposition, we refer the reader to Bourbaki [1] or J. Kelley's book [5].

Proof of the Compact Pseudometric Theorem. Since $q_1 \neq q_2$, there exists a formula $\varphi(\bar{x}, \bar{y})$, a tuple $\bar{e} \in E$, and an approximation ψ of φ such that

(*) $$\varphi(\bar{x}, \bar{e}) \in q_1, \qquad \psi(\bar{x}, \bar{e}) \notin q_2.$$

Now, given formulas

$$\sigma_1(\bar{x}, \bar{y}) < \sigma'_1(\bar{x}, \bar{y})$$
$$\vdots$$
$$\sigma_k(\bar{x}, \bar{y}) < \sigma'_k(\bar{x}, \bar{y}),$$

we define a set of pairs

$$V[\sigma_1, \sigma'_1, \dots, \sigma_k, \sigma'_k] \subseteq (\mathcal{B}_N)^n \times (\mathcal{B}_N)^n$$

as follows: $V[\sigma_1, \sigma'_1, \dots, \sigma_k, \sigma'_k]$ is the set of all pairs $(\bar{c}, \bar{c}')$ such that for every nonforking extension $\mathbf{r}(\bar{y})$ of $\mathrm{tp}(\bar{e}/A)$ and every $i = 1, \dots, k$,

$$\sigma_i(\bar{c}, \bar{y}) \in \mathbf{r} \quad \text{implies} \quad \sigma'_i(\bar{c}', \bar{y}) \in \mathbf{r}$$
$$\sigma_i(\bar{c}', \bar{y}) \in \mathbf{r} \quad \text{implies} \quad \sigma'_i(\bar{c}, \bar{y}) \in \mathbf{r}.$$

Step 1. *The family*

$$\mathfrak{V} = \{ V[\sigma_1, \sigma'_1, \dots, \sigma_k, \sigma'_k] \mid \sigma_i < \sigma'_i, \ k < \omega \}$$

is a base for uniform structure on $(\mathcal{B}_N)^n$ (in the sense of Chapter 6 of [5]).

Proof of Step 1. Since the definition of $V[\sigma_1, \sigma'_1, \dots, \sigma_k, \sigma'_k]$ is symmetric in $\bar{c}$ and $\bar{c}'$, each $V \in \mathfrak{V}$ is a symmetric subset of $(\mathcal{B}_N)^n \times (\mathcal{B}_N)^n$. It is also clear that each $V \in \mathfrak{V}$ contains the diagonal of $(\mathcal{B}_N)^n$. Now we check the remaining conditions.

For every $V_1, V_2 \in \mathfrak{V}$ there exists $W \in \mathfrak{V}$ such that $W \subseteq V_1 \cap V_2$: Let

$$V_1 = V[\sigma_1, \sigma'_1, \dots, \sigma_k, \sigma'_k],$$
$$V_2 = V[\tau_1, \tau'_1, \dots, \tau_l, \tau'_l].$$

If

$$W = V[\sigma_1, \sigma_1', \dots, \sigma_k, \sigma_k', \tau_1, \tau_1', \dots, \tau_l, \tau_l'],$$

then $W \subseteq V_1 \cap V_2$.

For every $V \in \mathfrak{V}$ there exists $W \in \mathfrak{V}$ such that $W \circ W \subseteq V$: Suppose

$$V = V[\sigma_1, \sigma_1', \dots, \sigma_k, \sigma_k'].$$

Choose L-formulas $\tau_1(\bar{x}, \bar{y}), \dots, \tau_k(\bar{x}, \bar{y})$ such that

$$\sigma_1(\bar{x}, \bar{y}) < \tau_1(\bar{x}, \bar{y}) < \sigma_1'(\bar{x}, \bar{y})$$

$$\vdots$$

$$\sigma_k(\bar{x}, \bar{y}) < \tau_k(\bar{x}, \bar{y}) < \sigma_k'(\bar{x}, \bar{y}).$$

Let

$$W = V[\sigma_1, \tau_1, \dots, \sigma_k, \tau_k, \tau_1, \sigma_1', \dots, \tau_k, \sigma_k'].$$

Then $W \circ W \subseteq V$. ⊣

Since, by assumption, the language is countable, $\mathfrak{V}$ is countable. Take subsets $\{ V_m \mid m < \omega \}$ of $X \times X$ such that:

- $V_0 = X \times X$;
- V_m is a vicinity of $\mathfrak{V}$, for each $m > 0$;
- $V_{m+1} \circ V_{m+1} \circ V_{m+1} \subseteq V_m$ for every m;
- Every vicinity of $\mathfrak{V}$ contains some V_m.

Let now ρ be a pseudometric on $\mathcal{B}_N$ which corresponds to $\{ V_m \mid m < \omega \}$ as in Proposition 5.2. Then $\mathfrak{V}$ is a base for the uniform structure of ρ.

Step 2. *ρ is definable over A.*

Proof of step 2. We just need to show that $f(\rho) = \rho$ for every A-automorphism f of $\mathfrak{C}$. Let f be such an automorphism. By Isomorphism and Uniqueness,

$$\{ f(\mathbf{r}) \mid \mathbf{r} \text{ is a nonforking extension of } \mathrm{tp}(\bar{e}/A) \} = \{ \mathbf{r} \mid \mathbf{r} \text{ is a nonforking extension of } \mathrm{tp}(\bar{e}/A) \}$$

Hence, for $(\bar{c}, \bar{c}') \in (\mathcal{B}_N)^n \times (\mathcal{B}_N)^n$, we have $(\bar{c}, \bar{c}') \in V$ if and only if $(\bar{c}, \bar{c}') \in f(V)$, i.e., $f(V) = V$. Thus, $f(V_m) = V_m$ for every m, and $f(\rho) = \rho$ by Proposition 5.2. ⊣

Step 3. *$(\mathcal{B}_N)^n$ is closed with respect to ρ.*

Proof of Step 3. Suppose that there exists a sequence $(\bar{c}_m)$ in $(\mathcal{B}_N)^n$ such that $\bar{c}_m \to \bar{c}$ as $m \to \infty$. We show that $\bar{c} \in (\mathcal{B}_N)^n$. Take $\epsilon > 0$, and let

$$V = V[\ \|\bar{x}\| \le N,\ \|\bar{x}\| \le N + \epsilon\ ].$$

For for large m, we have $(\bar{c}_m, \bar{c}) \in V$, so $\|\bar{c}\| \le N + \epsilon$. Since ϵ is arbitrary, we must have $\bar{c} \in (\mathcal{B}_N)^n$. ⊣

Step 4. *$(\mathcal{B}_N)^n$ is compact with respect to ρ.*

Proof of step 4. By Step 3, we only need to prove that $(\mathcal{B}_N)^n$ is precompact with respect to ρ. Suppose that this is not the case. Then there exists $\delta > 0$ and a sequence $(\bar{c}_m)$ in $(\mathcal{B}_N)^n$, such that $\rho(\bar{c}_i, \bar{c}_j) > \delta$, for $i < j < \omega$.

Let $\kappa = (2^{\aleph_0})^+$. Since ρ is definable over A, the compactness theorem provides a sequence $(\bar{d}_i \mid, i < \kappa)$ such that $\rho(\bar{d}_i, \bar{d}_j) > \delta/2$, for $i < j < \kappa$. Since $\mathfrak{V}$ is a base for the uniform structure of ρ, there exists $V \in \mathfrak{V}$ such that $(\bar{d}_i, \bar{d}_j) \notin V$, for $i < j < \kappa$. By refining the sequence $(\, \bar{d}_i \mid i < \kappa\,)$, we can assume that V is of the form $V[\sigma, \sigma']$, where σ and σ' are positive bounded formulas. Thus, for each pair (i, j) with $i < j < \kappa$ there exists a type $\mathbf{r}_{i,j}(\bar{y})$ such that

(i) $\mathbf{r}_{i,j}$ is a nonforking extension of $\mathrm{tp}(\bar{e}/A)$
(ii) $\sigma(\bar{d}_i, \bar{y}) \in \mathbf{r}_{i,j}$ and $\sigma'(\bar{d}_j, \bar{y}) \notin \mathbf{r}_{i,j}$, or else, $\sigma(\bar{d}_j, \bar{y}) \in \mathbf{r}_{i,j}$ and $\sigma'(\bar{d}_i, \bar{y}) \notin \mathbf{r}_{i,j}$.

For $\alpha < \kappa$, define $X_\alpha = \{\, \mathbf{r}_{i,j} \mid \sigma(\bar{d}_\alpha, \bar{y}) \in \mathbf{r}_{i,j}\,\}$. From (ii) above, we see that $X_\alpha \neq X_\beta$ for $\alpha < \beta < \kappa$. But this is impossible because, by (i) and Boundedness, there can be at most 2^κ such X_α's. $\dashv$

Now we finish the proof of the proposition. Take $\bar{d}_1$ and $\bar{d}_2$ such that $\bar{d}_1$ realizes q_1, $\bar{d}_2$ realizes q_2, and $\bar{d}_1$ and $\bar{d}_2$ are independent over E. By Proposition 4.4, $\mathrm{tp}(\bar{d}_1 \frown \bar{d}_2/E)$ does not fork over A. By Symmetry and Monotonicity,

(**) $$\mathrm{tp}(\bar{e}/A \cup \bar{d}_1 \cup \bar{d}_2) \text{ does not fork over } A.$$

Let $V = V[\varphi, \psi]$ (where φ and ψ are the formulas chosen in the second line of the proof). From $(*)$ and $(**)$, we conclude $(\bar{d}_1, \bar{d}_2) \notin V$. Thus, there exists $\delta > 0$ such that

(†) $$\rho(\bar{d}_1, \bar{d}_2) > \delta.$$

Take $M < N$ such that $\bar{a}, \bar{b} \in (\mathcal{B}_M(E))^n$. By Steps 2 and 4, , $(\,(\mathcal{B}_M)(E)\,)^n$ is precompact with respect to ρ. Hence, there exist $\bar{d}_1', \bar{d}_2' \in E$ such that

(‡) $$\rho(\bar{d}_1, \bar{d}_1') < \frac{\delta}{6}, \qquad \rho(\bar{d}_2, \bar{d}_2') < \frac{\delta}{6}.$$

Let now $\bar{c}_1$ and $\bar{c}_2$ be realizations of q_1 and q_2. Then $\mathrm{tp}(\bar{c}_1/E) = \mathrm{tp}(\bar{d}_1/E)$ and $\mathrm{tp}(\bar{c}_2/E) = \mathrm{tp}(\bar{d}_2/E)$. Thus,

$$\rho(\bar{c}_1, \bar{d}_1') < \frac{\delta}{6}, \qquad \rho(\bar{c}_2, \bar{d}_2') < \frac{\delta}{6}.$$

From (†) and (‡), we conclude $\rho(\bar{c}_1, \bar{c}_2) > \frac{\delta}{3}$. This finishes the proof of the theorem. $\dashv$

5.3. Definition. Let $\bar{a}$ and $\bar{b}$ be n-tuples in $\mathfrak{C}$. We say that $\bar{a}$ and $\bar{b}$ *have the same strong type over* A, and write $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$, if $\rho(\bar{a}, \bar{b}) = 0$, whenever ρ is a pseudometric defined on some $(\mathcal{B}_N)^n$, with $N > \|\bar{a}\|, \|\bar{b}\|$, such that

(1) ρ is definable over A;
(2) $(\mathcal{B}_N)^n$ is compact with respect to ρ.

5.4. Theorem. *Let $\bar{a}$ and $\bar{b}$ be n-tuples in $\mathfrak{C}$. The following conditions are equivalent.*

(1) $\mathrm{stp}\,\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$.
(2) $\mathrm{tp}(\bar{a}/E) = \mathrm{tp}(\bar{b}/E)$, *for every model E containing A.*
(3) $\mathrm{tp}(\bar{a}/E) = \mathrm{tp}(\bar{b}/E)$, *for some model E containing A.*

Proof. (1) $\Rightarrow$ (2) is the Compact Pseudometric Theorem. (2) $\Rightarrow$ (3) is clear. We prove (3) $\Rightarrow$ (1).

Take a model E containing A such that $\mathrm{tp}(\bar{a}/E) = \mathrm{tp}(\bar{b}/E)$, and suppose that $\mathrm{stp}(\bar{a}/A) \neq \mathrm{stp}(\bar{b}/A)$. Find $N > \|\bar{a}\|, \|\bar{b}\|$ and a pseudometric ρ on $(\mathcal{B}_N)^n$ such that

(i) ρ is definable over A;
(ii) $(\mathcal{B}_N)^n$ is compact with respect to ρ;
(iii) $\rho(\bar{a}, \bar{b}) > \alpha$, for some $\alpha > 0$.

Claim. *There exists K with $\|\bar{a}\|, \|\bar{b}\| < K < N$ satisfying the following property. For every $\epsilon > 0$ there exist $\bar{c}_1, \bar{c}_2 \in \mathcal{B}_K(E)$ such that $\rho(\bar{a}, \bar{c}_1) \leq \epsilon$ and $\rho(\bar{b}, \bar{c}_2) \leq \epsilon$.*

Proof of the Claim. We prove the assertion for $\bar{a}$ (this is clearly sufficient). Suppose, by way of contradiction, that there exist K with $\|\bar{a}\| < K < N$ and $\epsilon > 0$ such that the ρ-ball of radius ϵ around $\bar{a}$ does not intersect $\mathcal{B}_K(E)$. By (ii), there exists a positive integer t such that there are at most $\bar{c}_1, \ldots, \bar{c}_t \in \mathcal{B}_N(E)$ with $\rho(\bar{c}_i, \bar{c}_j) \geq \frac{\epsilon}{2}$ for $1 \leq i < j \leq t$.

Now, for M and δ with $K < M < N$ and $0 < \frac{\epsilon}{2} < \delta < \epsilon$, let $s(M, \delta)$ be the largest integer m such that there exist $\bar{c}_1, \ldots, \bar{c}_m \in \mathcal{B}_M(E)$ with $\rho(\bar{c}_i, \bar{c}_j) \geq \delta$ for $1 \leq i < j \leq m$. If $K < M < M' < N$ and $\frac{\epsilon}{2} < \delta < \delta' < \epsilon$, then $s(M, \delta') \leq s(M', \delta) \leq t$, so there exist M_0 and δ_1 such that $s(M, \delta) = s(M_0, \delta_0)$ for M and δ with $M_0 \leq M < N$ and $\frac{\epsilon}{2} < \delta \leq \delta_1 < \epsilon$.

Take M_1 and δ_0 with $M_0 < M_1 < N$ and $\frac{\epsilon}{2} < \delta_0 \leq \delta_1 < \epsilon$. Let $s = s(M_0, \delta_1)$. Take $\bar{c}_1, \ldots, \bar{c}_s \in \mathcal{B}_{M_0}(E)$ with $\rho(\bar{c}_i, \bar{c}_j) \geq \delta_1$ for $1 \leq i < j \leq s$. By our contradiction hypothesis, $\rho(\bar{a}, \bar{a}_i) \geq \epsilon > \delta_1$ for $i = 1, \ldots, s$. Hence,

$$\models_{\mathcal{A}} \exists \bar{x}_1, \ldots, \bar{x}_s, \bar{y} \Big(\bigwedge_{1\leq i\leq s} \|\bar{x}_i\| \leq M_0 \ \wedge \ \|\bar{y}\| \leq M_0 \ \wedge \\ \bigwedge_{1\leq i<j\leq s} \rho(\bar{x}_i, \bar{x}_j) \geq \delta_1 \ \wedge \ \bigwedge_{1\leq i\leq s} \rho(\bar{y}, \bar{x}_i) \geq \delta_1 \Big).$$

Since $E \prec_{\mathcal{A}} \mathfrak{E}$,

$$E \models \exists \bar{x}_1, \ldots, \bar{x}_s, \bar{y} \Big(\bigwedge_{1\leq i\leq s} \|\bar{x}_i\| \leq M_1 \ \wedge \ \|\bar{y}\| \leq M_1 \ \wedge \\ \bigwedge_{1\leq i<j\leq s} \rho(\bar{x}_i, \bar{x}_j) \geq \delta_0 \ \wedge \ \bigwedge_{1\leq i\leq s} \rho(\bar{y}, \bar{x}_i) \geq \delta_0 \Big).$$

But then, since the ρ-ball of radius δ_0 around $\bar{a}$ does not intersect $\mathcal{B}_{M-1}(E)$, we have $s(M_1, \delta_0) \geq s + 1$, which contradicts the choice of M_1 and δ_0 ⊣

Now we conclude the proof of the theorem. By the claim, there exist $\bar{c}_1, \bar{c}_2 \in E$ such that $\rho(\bar{a}, \bar{c}_1) \leq \frac{\alpha}{3}$ and $\rho(\bar{b}, \bar{c}_2) \leq \frac{\alpha}{3}$. The formula $\|\bar{x} - \bar{c}_1\| \leq \frac{\alpha}{3}$ is in $\mathrm{tp}(\bar{a}/E)$, but not in $\mathrm{tp}(\bar{b}/E)$. Thus, $\mathrm{tp}(\bar{a}/E) \neq \mathrm{tp}(\bar{b}/E)$. ⊣

6. Forking extensions

In first-order model theory, an extension of a type to a larger set of parameters can be either forking or nonforking. In Banach space model theory, a finer analysis of forking is needed. Namely, when one extension is forking, we need to specify "how much" it forks. For instance, suppose that the uniform structure $\mathfrak{U}$ is metrizable. Then an extension of a type is ϵ-forking if it is at least ϵ-away from any nonforking extension.

In this section, rather than developing a quantitative theory of forking, we prove the necessary facts about forking extensions to prove the Spectrum Theorem in Section 8.

We begin by proving the following fact about uniform structures which is interesting in its own right. It states that the neighborhoods of the topology given by a uniform structure on the space of types are "uniformly definable".

Proposition 6.1. *For every vicinity U there is a vicinity $W \subseteq U$ with the following property. For every type $p \in S_n(B)$ there exists a set of $L(B)$-formulas $\Sigma(\bar{x})$ such that*

(1) If $q \in S_n(B)$ and $q \supseteq \Sigma$, then $(p, q) \in U$.

(2) If $(p, q) \in W$, then $q \supseteq \Sigma$.

Proof. Take a vicinity V corresponding to U as in (2-iii) of the definition of uniform structure [3]. In turn, take W corresponding to V in the same fashion. Then $W \subseteq V \subseteq U$.

Let

$$\Sigma_1(\bar{x}) = \{\, \psi'(\bar{x}, \bar{b}) \mid (\, \psi(\bar{x}, \bar{y}), \psi'(\bar{x}, \bar{y})\,) \in \mathcal{D}(V),\ \psi(\bar{x}, \bar{b}) \in p,\ \bar{b} \in \mathbb{Q}B \,\}$$

$$\Sigma_2(\bar{x}) = \{\, \mathrm{neg}(\psi(\bar{x}, \bar{b})) \mid (\, \psi(\bar{x}, \bar{y}), \psi'(\bar{x}, \bar{y})\,) \in \mathcal{D}(V),\ \mathrm{neg}(\psi'(\bar{x}, \bar{b})) \in p_+,\ \bar{b} \in \mathbb{Q}B \,\}.$$

Now define $\Sigma(\bar{x}) = \Sigma_1(\bar{x}) \cup \Sigma_2(\bar{x})$.

(1): Take a type $q(\bar{x}) \in S(B)$ extending Σ. We prove that $(p, q) \in U$. Fix

$$(\varphi(\bar{x}, \bar{y}), \varphi'(\bar{x}, \bar{y})) \in \mathcal{D}(U)$$

and $\bar{b} \in \mathbb{Q}B$. We show that for every $\varphi'' > \varphi'$,

(*) $\qquad \varphi(\bar{x}, \bar{b}) \in p \quad$ implies $\quad \varphi''(\bar{x}, \bar{b}) \in q$

(**) $\qquad \varphi(\bar{x}, \bar{b}) \in q \quad$ implies $\quad \varphi''(\bar{x}, \bar{b}) \in p.$

Fix $\varphi'' > \varphi'$. Find $(\psi, \psi') \in \mathcal{D}(V)$ such that $\varphi < \psi$ and $\psi' < \varphi''$.

Suppose $\varphi(\bar{x}, \bar{b}) \in p$. Then $\psi(\bar{x}, \bar{b}) \in p$ and by definition, $\psi'(\bar{x}, \bar{b}) \in \Sigma_1 \subseteq q$. Hence, $\varphi''(\bar{x}, \bar{b}) \in q$. This proves (*).

If $\varphi''(\bar{x}, \bar{b}) \notin p$, then $\mathrm{neg}(\varphi''(\bar{x}, \bar{b})) \in p$, so $\mathrm{neg}(\psi'(\bar{x}, \bar{b})) \in p_+$. By definition, $\mathrm{neg}(\psi(\bar{x}, \bar{b})) \in \Sigma_2 \subseteq q$. Hence $\varphi(\bar{x}, \bar{b}) \notin q$. This proves (**).

(2): Suppose $(p, q) \in W$. Then $(p, q) \in V$, so $q \supseteq \Sigma_1$. We prove that, also, $q \supseteq \Sigma_2$.

Suppose that $(\psi, \psi') \in \mathcal{D}(V)$ and $\mathrm{neg}(\psi'(\bar{x}, \bar{b})) \in p_+$ for some $b \in \mathbb{Q}B$. Then $\mathrm{neg}(\psi'(\bar{x}, \bar{b}))$ is an approximation of a formula in p, so there exists $\psi'' > \psi'$ such that $\mathrm{neg}(\psi''(\bar{x}, \bar{b})) \in p$. By the choice of W, there exists $(\chi, \chi') \in \mathcal{D}(W)$ such that $\psi < \chi$ and $\chi' < \psi''$. We have $\chi'(\bar{x}, \bar{b}) \notin p$. Then $\chi(\bar{x}, \bar{b}) \notin q$ (since $(p, q) \in W$), so $\mathrm{neg}(\psi(\bar{x}, \bar{b})) \in q$. ⊣

6.2. Definition. Let U be a vicinity. Let $A \subseteq B$, and $p \in S(B)$. We say that *p U-forks over A*, or that *p is an U-forking extension of $p|A$*, if $(p, q) \notin U$ whenever q is a nonforking extension of p over B.

Corollary 6.3. *Let $A \subseteq B$, and let W correspond to U as in Proposition 6.1. Then for every type $p(\bar{x}) \in S(B)$ there is a set of $L(B)$-formulas $\Phi_U(\bar{x})$ such that*

(1) If p does not U-fork over A, then $\Phi_U \cup p|A$ is consistent.

(2) If $q \supset \Phi_U \cup p|A$, then q does not fork over A and $(p, q) \in U$.

Proof. Let p be a nonforking extension of r over B. Let W and Σ correspond to p as in Proposition 6.1 and define $\Phi_U(\bar{x}) = \Sigma \cup \mathcal{N}_A(\bar{x})$. ⊣

Corollary 6.4. *Let $A \subseteq B$ and $p \in S(B)$. Then p forks over A if and only if p U-forks over A, for some vicinity U.*

Proof. $\Leftarrow$ is clear. To prove $\Rightarrow$, suppose that p does not U-fork over A, for any vicinity U. By Corollary 6.3, the set

$$p|A \cup \bigcap_{U \in \mathfrak{U}} \Phi_U$$

is consistent. Hence, there exists a type q such that q does not fork over A, and $(p, q) \in U$ for every vicinity U, i.e., $q = p$. ⊣

The following corollary is a refinement of the Open Map Theorem.

Corollary 6.5. *Let $A \subseteq B$, and let W correspond to U as in Proposition 6.1. Then, if $p \in S(B)$ and p U-forks over A, there exists a formula $\varphi \in p_+$ such that any extension of $\{\varphi\} \cup p|A$ must W-fork over A.*

Proof. Let Φ_U be as in Corollary 6.3. Then $p \cup \Phi_U$ is inconsistent, so there exists $\varphi \in p_+$ such that $\{\varphi\} \cup p|A \cup \Phi_U$ is inconsistent. By Corollary 6.3, any extension of $\{\varphi\} \cup p|A$ must W-fork over A. $\dashv$

6.6. Definition. Let $A \subseteq B$, and let $p \in S(B)$. If U is a vicinity, we say that p *definably U-forks over A*, or that *p is a definably U-forking extension of $p|A$*, if there exists a formula $\varphi \in p_+$ such that any type containing φ must U-fork over A. In this case we say that φ *defines* p as a U-forking extension of $p|A$.

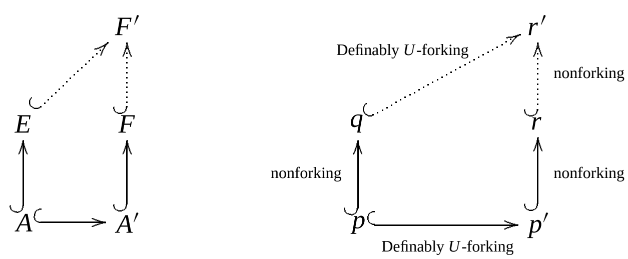
Corollary 6.5 says that for every vicinity U there exists a vicinity $W \subseteq U$ such that if every type which U-forks over A must W-fork over A definably.

Proposition 6.7. *Let $A \subseteq A'$. Suppose that $p(\bar{x}) \in S(A)$ and p' is a definably U-forking extension of p over A'. Suppose also that*

- *$E \supseteq A$;*
- *$F \supseteq A'$;*
- *q is a nonforking extension of p over E;*
- *r is a nonforking extension of q over F.*

Then, there exists

- *A model F' such that $F \prec_{\mathcal{A}} F'$;*
- *A nonforking extension r' of r over F';*
- *An elementary embedding $f : E \prec_{\mathcal{A}} F'$ such that r' is a definably U-forking extension of $f(p)$.*



Remarks.

(1) Suppose φ defines p' as a U-forking extension of p. Then φ defines any extension of q containing φ as a U-forking extension. Therefore, the conclusion that r' is a definably U-forking extension of $f(p)$ is redundant.
(2) The model F' can be taken with density$(F') \leq$ density$(E) +$ density(F).

Proof of Proposition 6.7. For each $c \in E$, choose a new name c'. By Proposition 10.2, $\mathrm{cl}(q) \subseteq \mathrm{cl}(r)$. Hence, the set

$$\Sigma(\bar{x}) = r(\bar{x}) \cup \mathcal{N}_F(\bar{x}) \cup \{\, \varphi(\bar{x}, \bar{c}') \mid \varphi(\bar{x}, \bar{c}) \in q_+ \,\}$$

is consistent . Let $(c'' \mid c \in E)$ be an interpretation of the constants $(c' \mid c \in E)$ such that the interpretation $c' \mapsto c''$ makes Σ consistent over $F \cup \{\, c'' \mid c \in E \,\}$. The lemma follows immediately by defining $f(c) = c''$. $\dashv$

7. Forking and superstability

7.1. Definition. Let U be a vicinity. Let $A_0 \subseteq A_1 \subseteq \dots$ be a countable chain of subsets of $\mathfrak{C}$. A *U-forking chain over* $(A_i \mid i < \omega)$ is a chain of types $p_0 \subseteq p_1 \subseteq \dots$ such that $p_j \in S(A_j)$ for each $j < \omega$, and p_j is a U-forking extension of p_i, for every $i < j$.

We say that the chain $(p_i \mid i < \omega)$ is *a definably U-forking chain* if p_j is a definably U-forking extension of p_i, for each $i < j$.

7.2. Lemma. *Suppose that $(p_i \mid i < \omega)$ is a definable U-forking chain over $(A_i \mid i < \omega)$. Suppose also that each A_i is separable. Then for each $i < \omega$ there exist $(A_{i,j} \mid j < \omega)$ and a definable U-forking chain $(p_{i,j} \mid j < \omega)$ over $(A_{i,j} \mid j < \omega)$ such that for each $j < \omega$,*

(1) $A_{i,j}$ is separable;
(2) $A_{i,0} = A_{i+1}$;
(3) $A_{i,j} \supseteq A_{i+j+1}$;
(4) $(\, p_{i,j}|A_{i,j+1}\,,\, p_{i+j+1}\,) \notin U$.

Proof. Fix $i < \omega$. We construct $(p_{i,j} \mid j < \omega)$ and $(A_{i,j} \mid j < \omega)$ such that

(1) $A_{i,j}$ is separable;
(2) $A_{i,0} = A_{i+1}$;
(3) $A_{i,j} \supseteq A_{i+j+1}$;
(4) $p_{i,j}$ is a nonforking extension of p_{i+j+1} over $A_{i,j}$.

This will prove the lemma.

Fix $i < \omega$. We define $A_{i,j}$ and $p_{i,j}$ by induction on j. Let $A_{i,0} = A_{i+1}$, and let $p_{i,0}$ be a nonforking extension of p_j over $A_{i,0}$. Suppose now that $A_{i,j}$ and $p_{i,j}$ have been defined, in order to define $A_{i,j+1}$ and $p_{i,j+1}$.

Take a separable model $F \supseteq A_{i+j+2}$ and a nonforking extension r of $p_{i,j+1}$ over F. Proposition 6.7 provides a separable extension F' of F and a nonforking extension r' of r over F' such that r' is a definably U-forking extension of $p_{i,j}$. We define $A_{i,j+1} = F'$ and $p_{i,j+1} = r'$. ⊣

The following proposition establishes the first connection between U-forking chains and stability.

Proposition 7.3. *Suppose that for some vicinity U, there exists a U-forking chain. Then, if κ is a cardinal κ such that $\kappa^{\aleph_0} > \kappa$, the theory T is κ-unstable with respect to $\mathfrak{U}$.*

Proof. Suppose that $(p_i \mid i < \omega)$ is a U-forking chain over $(A_i \mid i < \omega)$. Corollary 6.5 allows us to assume that $(p_i \mid i < \omega)$ is a definably U-forking chain, and that each A_i is separable. Iterative application of Proposition 6.7 yields separable sets $(\, A_s \mid s \in \kappa^{<\omega}\,)$ and types $(\, p_s \mid s \in \kappa^{<\omega}\,)$ such that for each $s \in \kappa^{<\omega}$,

(1) $(\, p_{s\frown i} \mid s \in \kappa^{<\omega}, i < \omega\,)$ is a U-forking chain over $(\, A_{s\frown i} \mid s \in \kappa^{<\omega}, i < \omega\,)$;
(2) $A_{s\frown i}$ is separable;
(3) $A_{s\frown i\frown 0} = A_{s\frown(i+1)}$;
(4) $A_{s\frown i\frown j} \supseteq A_{s\frown(i+j+1)}$;
(5) $(\, p_{s\frown i\frown j}|A_{s\frown(i+j)},\ p_{s\frown(i+j+1)}\,) \notin U$.

Let $A = \bigcup_{s\in\kappa^{<\omega}} A_s$. Then $\operatorname{card}(A) \leq \kappa$ by (2).

For each $\xi \in \kappa^{\omega}$, let p_ξ be an extension of $\bigcup_{\substack{s\in\kappa^{<\omega}\\ s\subset\xi}} p_s$ over A. Then $p_\xi \in S(A)$, for every $\xi \in \kappa^{\omega}$. We prove that $(p_\xi, p_\eta) \notin U$, for $\xi \neq \kappa$. This will show that T is not κ-stable with respect to $\mathfrak{U}$, because then the density of $S(A)$ with respect to $\mathfrak{U}$ is greater than κ (but $\operatorname{card}(A) \leq \kappa$).

Take distinct sequences $\xi, \eta \in \kappa^\omega$. Then there exist $s \in 2^{<\omega}$ and distinct integers i and j such that $s \frown i \subset \xi$ and $s \frown j \subset \eta$. We may assume $i < j$. Take k such that $s \frown i \frown k \subset \xi$.

By (5) above,

$$(p_{s\frown i \frown 0},\ p_{s\frown (i+1)}) \notin U.$$

But

$$p_{s\frown i\frown 0} \subseteq p_{s\frown i\frown k} \subseteq p_\xi$$

and

$$p_{s\frown (i+1)} \subseteq p_{s\frown j} \subseteq p_\eta.$$

Hence, $(p_\xi, p_\eta) \notin U$. ⊣

7.4. Definition. A theory T is *superstable* with respect to $\mathfrak{U}$ if T is κ-stable with respect to $\mathfrak{U}$, for every cardinal κ with $\kappa \geq 2^{\aleph_0}$.

7.5. Theorem. *The following conditions are equivalent.*

(1) T is superstable.
(2) For any vicinity U, there is no U-forking chain.
(3) If $p \in S(A)$ and U is a vicinity, there is a finite tuple $\bar{a} \in A$ such that p does not U-fork over $\bar{a}$.
(4) If $A \subseteq \mathfrak{C}$, the set of n-types that do not fork over some finite subset of A is $\mathfrak{U}$-dense in $S(A)$.

Proof. (1) ⇒ (2): Suppose that there is a U-forking chain, for some vicinity U. Take a cardinal κ such that $\kappa > 2^{\aleph_0}$ and $\kappa^{\aleph_0} > \kappa$. By Proposition 7.3, T is not κ-stable with respect to $\mathfrak{U}$. Hence, T is not superstable.

(2) ⇒ (3): Let $p \in S(A)$, and suppose that p U-forks over every finite subset of A. Let W correspond to U as in Corollary 6.5. Inductively, we find a chain of finite subsets of A, $A_0 \subseteq A_1 \subseteq \dots$ such that $p|A_j$ is a W-forking extension of $p|A_i$, for $i < j$.

(3) ⇒ (4): Fix a vicinity U and a type $p \in S(A)$. Find $\bar{a} \in A$ such that p does not U-fork over $\bar{a}$. By the definition of U-forking, there exists a nonforking extension q of $p|\bar{a}$ over A, such that $(p, q) \in U$.

(4) ⇒ (1): Let $\kappa \geq 2^{\aleph_0}$, and let A be a set of cardinality κ. For each $\bar{a} \in A$ there are at most $2^{\aleph_0}$ types over $\bar{a}$, and each of them has $2^{\aleph_0}$ nonforking extensions over A (by Boundedness). Therefore, there are at most $\kappa \cdot 2^{\aleph_0} = \kappa$ types over A which $2^{\aleph_0}$ do not fork over some finite subset of A. Thus, density$(S(A)) \leq \kappa$. ⊣

8. Stability spectrum

In this section, the terms "stable" and "density character" are to be taken with respect to our fixed uniform structure $\mathfrak{U}$.

8.1. Theorem. *Suppose that $\mathfrak{U}$ is metrizable. Then, for any theory T, one of the following conditions must be true:*

(1) T is not stable;
(2) T is κ stable for every infinite cardinal κ;
(3) T is κ-stable if and only if $\kappa \geq 2^{\aleph_0}$;
(4) T is κ-stable if and only if $\kappa^{\aleph_0} = \kappa$.

Proof. Suppose that T is superstable. Then, either T is ω-stable, and (2) holds, or density$(S(E)) > \aleph_0$ for some separable model E. In this case, the reader can show that, in fact, density$(S(E)) \geq 2^{\aleph_0}$, so T is not κ-stable for $\aleph_0 \leq \kappa < 2^{\aleph_0}$. Hence, (3) holds.

Now suppose that T is stable but not superstable in order to prove (4). Take a cardinal κ such that $\kappa < \kappa^{\aleph_0}$. By Theorem 7.5, there exists a U forking chain, for some vicinity U. But then Proposition 7.3 implies that T is not κ-stable.

If $\kappa = \kappa^{\aleph_0}$ and density$(E) \leq \kappa$, then card$(S(E)) \leq \kappa$ (for there are at most $\kappa^{\aleph_0} = \kappa$ definition schemata for types over E), so T is κ-stable. ⊣

9. Examples

In this section we exhibit two examples. The first example is of a theory which is ω-stable but not superstable with respect to the metric d. The second example is of a theory which is stable but not superstable with respect to d.

At this point we should remind the reader that we already know examples of theories that are superstable but not ω-stable with respect to the metric D, namely, any Banach space theory which is ω-stable with respect to d (e.g., Hilbert spaces, ℓ_p spaces, for $1 \leq p < \infty$) is superstable but not ω-stable with respect to D (see [3]).

Some of the computations with positive bounded formulas involved in the examples of this section are rather lengthy, and we have omitted them for the sake of clarity.

Example 1. A superstable, not ω-stable theory: . Let H be an infinite dimensional Hilbert space. For each $s \in 2^{<\omega}$, find an infinite dimensional subspace H_s of H, such that

- $H_\emptyset = H$;
- $H_s = H_{s\frown 0} + H_{s\frown 1}$;
- $H_{s\frown 0} \perp H_{s\frown 1}$.

For each $s \in 2^{<\omega}$, we let

$$\rho_k(x) = \text{distance from } x \text{ to } \bigcup_{\ell(s)\leq k} H_s,$$

and $\mathbf{H} = (\, H, \rho_k \mid k < m \,)$.

Let T_m be the positive bounded theory of the structure $\mathbf{H}$. Suppose that $\mathbf{E} = (\, E, \wp_k \mid k < m \,)$ is a model of T_m. Then E is a Hilbert space. Also, for each $k < m$ and each $s \in 2^k$ there exists an infinite dimensional subspace E_k of E, such that

- $E_\emptyset = E$;
- $E_s = E_{s\frown 0} + E_{s\frown 1}$;
- $E_{s\frown 0} \perp E_{s\frown 1}$;

and

$$\wp_k(x) = \text{distance from } x \text{ to } \bigcup_{\ell(s)\leq k} E_s.$$

Thus, if $\mathbf{E}_1$ and $\mathbf{E}_2$ are separable models of T_m and $\bar{a} \in E_1$, $\bar{b} \in E_2$ have the same quantifier-free type, there is an isomorphism from $\mathbf{E}^1$ onto $\mathbf{E}^2$ carrying $\bar{a}$ to $\bar{b}$. This implies that T_m admits quantifier elimination (see [2]).

Let now $T = \bigcup_{m<\omega} T_m$. By the preceding argument, T is complete and admits quantifier elimination. We show that T is superstable but not ω-stable with respect to the metric d.

Fix a model $\mathbf{E} = (E, \wp_k \mid k < \omega)$ of T, of density character κ. We prove that the d-density character of $S_1(E)$ is $\max\{\, 2^{\aleph_0}, \kappa \,\}$.

For $s \in 2^{<\omega}$, find a subspace E_s of E as above. Let $\mathfrak{E}$ be a $2^{\aleph_0}$-saturated extension of $\mathbf{E}$. For each $\xi \in 2^\omega$ there exists an infinite dimensional subspace E_ξ of $\mathfrak{E}$, such that

- $E_s \subseteq E_\xi$, if $s \subset \xi$;
- $\mathfrak{E} = \sum_{\xi \in 2^\omega} E_\xi$;
- $E_\xi \perp E_\eta$, if $\xi \neq \eta$.

For each $\xi \in 2^\omega$, take $x_\xi \in E_\xi \setminus E$ with $\|x_\xi\| = 1$. Then, if $\xi \neq \eta$, the d-distance between $\text{tp}(x_\xi/E)$ and $\text{tp}(x_\eta/E)$ is $\sqrt{2}$. This shows that the d-density character of $S_1(E)$ is at least $2^{\aleph_0}$.

If $x, y \in \mathfrak{E} \setminus E$ and $x, y \in E_\xi$ for some $\xi \in 2^\omega$, then $\text{tp}(x/E) = \text{tp}(y/E)$. Hence, for purposes of counting types over E, we may assume that $E_\xi \setminus E$ is separable for each $\xi \in 2^\omega$. Let A_ξ be a dense subset of $E_\xi \setminus E$. Then, the set

$$\{\, \text{tp}(c/E) \mid c \in \sum_{\xi \in 2^\omega} A_\xi \,\}$$

is d-dense in the set of nonrealized types over A. But the above set has cardinality $\max\{\, 2^{\aleph_0}, \kappa \,\}$. Thus, the set of nonrealized 1-types over E (and hence the set of all 1-types over E) has density character $\max\{\, 2^{\aleph_0}, \kappa \,\}$.

Example 2. A stable, not superstable theory: . Let H be an infinite dimensional Hilbert space. For each $s \in \omega^{<\omega}$ let H_s be an infinite dimensional subspace of H such that

- $H_\emptyset = H$;
- $H_s = \sum_{i<\omega} H_{s\frown i}$;
- $H_{s\frown i} \perp H_{s\frown j}$, if $i < j$.

For each $s \in \omega^{<\omega}$, let

$$\rho_k(x) = \text{distance from } x \text{ to } \bigcup_{\ell(s)\leq k} H_s.$$

Let $\mathbf{H} = (\, H, \rho_k \mid k < m \,)$, and let T be the positive bounded theory of $\mathbf{H}$.

Let $\mathbf{E} = (\, E, \wp_k \mid k < \omega \,)$ be a model of T of density character κ. Then E is a Hilbert space, and there exist cardinals $\lambda_k \leq \kappa$ $(k < \omega)$ with the following property. For each $s \in (\Pi_{k<\omega}\lambda_k)^{<\omega}$ there exists an infinite dimensional subspace E_s of E, such that

- $E_\emptyset = E$;
- $E_s = \sum_{m\in\omega} E_{s\frown m}$;
- $E_{s\frown k} \perp E_{s\frown m}$, if $k < m$;

and

$$\wp_k(x) = \text{distance from } x \text{ to } \bigcup_{\ell(s)\leq k} E_s.$$

Arguing as in the preceding example, we see that the d-density character of $S_1(E)$ is $\kappa^{\aleph_0}$. Thus, T is κ-stable if and only if $\kappa^{\aleph_0} = \kappa$.

10. Appendix: Forking and classes

Recall from [3] that a positive bounded L-formula $\varphi(\bar{x}, \bar{y})$ is *represented* in a type $p \in S(B)$ if there exists $\bar{b} \in B$ such that $\varphi(\bar{x}, \bar{b}) \in p$, and *almost represented* in p if every approximation of φ is represented in p.

10.1. Definition. If p is a type, the *class* of p, denoted $\text{cl}(p)$, is the set of L-formulas that are almost represented in p.

Proposition 10.2. *Let $A \subseteq E$ and $p \in S(E)$. The following conditions are equivalent*

(1) p does not fork over A.
(2) If q is an extension of $p|A$ over a model containing A, then $\mathrm{cl}(p) \subseteq \mathrm{cl}(q)$.

Proof. (1) $\Rightarrow$ (2): Let q be an extension of p over a model F containing A, and suppose that $\mathrm{cl}(p) \not\subseteq \mathrm{cl}(q)$. Take a formula $\varphi(\bar{x}, \bar{y})$ in $\mathrm{cl}(p) \setminus \mathrm{cl}(q)$ and find $\psi > \varphi$ such that ψ is not represented in q. Take now φ' such that $\varphi < \varphi' < \psi$.

Let $\bar{c}_1$ be a realization of p, and $\bar{c}_2$ be a realization of q. Find an A-automorphism f of $\mathfrak{E}$, with $f(\bar{c}_1) = \bar{c}_2$. Since φ' is represented in p, there exists $\bar{b} \in E$ such that

(i) $\models \varphi'(\bar{c}_1, \bar{b})$, and
(ii) $\models \varphi'(\bar{c}_2, f(\bar{b}))$.

Since ψ is not represented in q, we have $F \subseteq \mathrm{neg}(\psi(\bar{c}_2, \mathfrak{E}))$; thus, by definition, $\mathrm{neg}(\psi(\bar{c}_2, \bar{y})) \in \mathcal{N}_A(\bar{y})$. Hence, by (ii) above, $\mathrm{tp}(f(\bar{b})/A \cup \bar{c}_2)$ forks over A. By Symmetry, $\mathrm{tp}(\bar{c}_2/A \cup f(\bar{b}))$ forks over A. But then, $\mathrm{tp}(\bar{c}_1/A \cup \bar{b})$ forks over A, which is impossible, since $\bar{b} \in E$ and $\mathrm{tp}(\bar{c}_1/E)$ does not fork over A.

(2) $\Rightarrow$ (1): Let $\mathbf{p}(\bar{x})$ be an heir of p over $\mathfrak{E}$. We claim that $\mathbf{p}$ does not fork over A. This will prove that p does not fork over A.

Let F be a model containing A. We have $\mathrm{cl}(\mathbf{p}) \subseteq \mathrm{cl}(p)$, since $\mathbf{p}$ is the heir of p. By (2), we also have $\mathrm{cl}(p) \subseteq \mathrm{cl}(\mathbf{p}|F)$. Hence, $\mathrm{cl}(\mathbf{p}) \subseteq \mathrm{cl}(\mathbf{p}|F)$. Since F is arbitrary, $\mathbf{p}$ does not fork over A, by Proposition 3.5. $\dashv$

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