STABLE BANACH SPACES AND BANACH SPACE STRUCTURES, I: FUNDAMENTALS

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ABSTRACT. We study model theoretical stability for Banach spaces and structures based on Banach spaces, e.g., Banach lattices or C^*-algebras. We prove that a theory is stable if and only if the following condition is true in every model E of the theory: If $(\bar{a}_m)$ and $(\bar{b}_n)$ are bounded sequences in E^k and E^l (respectively) and $\mathcal{R}\colon E^k \times E^l \to \mathbb{R}$ is definable, then there exist subsequences $(\bar{a}_{m_i})$ and $(\bar{b}_{n_j})$ such that

$$\lim_{i\to\infty}\lim_{j\to\infty}\mathcal{R}(\bar{a}_{m_i},\bar{b}_{n_j}) = \lim_{j\to\infty}\lim_{i\to\infty}\mathcal{R}(\bar{a}_{m_i},\bar{b}_{n_j}).$$

1. INTRODUCTION

A framework for the model theoretical analysis of various structures from functional analysis was introduced in the monograph [6]. The class of structures under consideration includes Banach spaces as well as further structures from functional analysis which are based on Banach spaces e.g., C^*-algebras. The model theoretical language of [6] provides tools to prove new results for important classes of structures from analysis.

In this series of papers we use the general framework of [6] to develop the theory of stable Banach space structures. Rich classes of Banach space are stable, e.g., the spaces L_p, for $1 \leq p < \infty$. In this, the first paper of the series, we introduce the basic concepts of stability, and show Banach space theoretical counterparts of familiar facts from classical stability theory, (e.g., a theory is stable if every type is definable). For this basic material we follow the order of exposition of [10].

In Section 9, we prove the following result:

Theorem. *A theory is stable if and only if the following condition is true in every model E of the theory: If $(\bar{a}_m)$ and $(\bar{b}_n)$ are bounded sequences in E^k and E^l (respectively) and $\mathcal{R}\colon E^k \times E^l \to \mathbb{R}$ is definable, then there exist subsequences $(\bar{a}_{m_i})$ and $(\bar{b}_{n_j})$ such that*

$$\lim_{i\to\infty}\big(\lim_{j\to\infty}\mathcal{R}(\bar{a}_{m_i},\bar{b}_{n_j})\big) = \lim_{j\to\infty}\big(\lim_{i\to\infty}\mathcal{R}(\bar{a}_{m_i},\bar{b}_{n_j})\big).$$

In particular, in a stable Banach space, the above condition holds for the real-valued function $\mathcal{R}(x,y) = \|x+y\|$. This fact imposes rather strong conditions on the geometry of stable Banach spaces: every such space contains the sequence space ℓ_p, for some $1 \leq p < \infty$, almost isometrically. See [9]. (See also the remarks concluding Section 9.)

In forthcoming papers, we will concentrate on deeper aspects of the theory, as well as direct applications to functional analysis.

We assume that the reader is familiar with the basic machinery of [6]. In particular, we assume familiarity with the notions of Banach space structure, positive bounded formula, and approximate satisfaction $\models_{\mathcal{A}}$. The terms "formula", "theory" and "type" are used

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as abbreviations of "positive bounded formula", "positive bounded theory", and "positive bounded type", respectively. If φ and ψ are positive bounded formulas, we write $\varphi < \psi$ to indicate that ψ is an approximation of φ.

The letter T denotes a fixed complete positive bounded theory over a countable language L. All the models considered are models of T. We assume that these models are approximately elementary submodels of some large, saturated model $\mathfrak{C}$. If $\bar{a} \in \mathfrak{C}$, $\|\bar{a}\|$ is taken as an abbreviation of $\max_{1\leq i\leq n} \|a_i\|$. (In other words, we equip the finite powers of $\mathfrak{C}$ with an ℓ_∞-norm.) If A is a subset of $\mathfrak{C}$ and $N \geq 0$, we denote the set of elements of A of norm $\leq N$ by $\mathcal{B}_N(A)$. When $N = 1$, we write simply $\mathcal{B}(A)$. By the $\aleph_0$-saturation of $\mathfrak{C}$, we have $\mathfrak{C} \models_{\mathcal{A}} \varphi(\bar{a})$ if and only if $\mathfrak{C} \models \varphi(\bar{a})$ for every positive bounded formula φ and every $\bar{a} \in \mathfrak{C}$. If $\mathfrak{C} \models_{\mathcal{A}} \varphi(\bar{a})$, we write simply $\models_{\mathcal{A}} \varphi(\bar{a})$, omitting $\mathfrak{C}$. If A is a subset of $\mathfrak{C}$, we denote by $\mathbb{Q}A$ the set of rational multiples of A.

We consider only complete, positive bounded types which are consistent with T. The norm of an n-type p, denoted $\|p\|$, is the norm of any n-tuple realizing p. If A is a subset of a model E, we denote by $L(A)$ the result of expanding the language L with constants and appropriate norm bounds for the elements of A, and $T(A)$ is the theory of $\mathfrak{C}$ in $L(A)$. The set of n-types over A is denoted $S_n(A)$, and $S(A) = \bigcup_{n<\omega} S_n(A)$.

2. Uniform Structures on the Space of Types

The space of types of a complete positive bounded theory T is endowed with various uniform topologies on it. Below are two of the most natural examples.

- *The metric d*. This is the result of "transferring" the norm topology from the models of T onto $S(T)$. If $p(\bar{x})$ and $q(\bar{x})$ are types, we let
$$d(p,q) = \inf\{\, \|\bar{b} - \bar{c}\| \mid \mathfrak{C} \models p(\bar{b}), q(\bar{c})\,\}.$$
It is easy to show that d is a metric on $S_n(A)$. We extend d to all of $S(T)$ by letting $d(p,q) = \infty$ when p and q are types in different sets of variables or over different sets of parameters.

  Notice that if $p(\bar{x})$ and $q(\bar{x})$ are types, then $d(p,q) = \alpha$ if and only if for every realization $\bar{c}$ of p there exists a realization $\bar{c}$ of q such that $\|\bar{b} - \bar{c}\| = \alpha$.
- *The Banach-Mazur metric D*. Suppose that the language contains no real-valued relations other than the norm, and that the only functions in addition to the vector space operations are constants. Thus, the L-structures are of the form $(E, c_i)_{i\in I}$, where E is a Banach space and the c_i's are constants. Let $\epsilon \geq 0$; a $(1+\epsilon)$-*isomorphism* between two L-structures $(E, c_i)_{i\in I}$ and $(F, c_i)_{i\in I}$ is a linear isomorphism $f : E \to F$ such that $f(c_i) = d_i$ for every $i \in I$, and $\|f\|, \|f^{-1}\| \leq 1+\epsilon$. If $p(\bar{x})$ and $q(\bar{x})$ are types, we let
$$D(p,q) = \inf\left\{ 1+\epsilon \;\middle|\; \begin{array}{c} \text{there is a } (1+\epsilon)\text{-isomorphism between} \\ (\mathfrak{C}, \bar{b}) \text{ and } (\mathfrak{C}, \bar{c}), \text{ and } \mathfrak{C} \models p(\bar{b}), q(\bar{c}) \end{array} \right\}.$$
  The function D itself is not a metric, but $\log(D)$ is. Nonetheless, it is the function D that is referred to as the *Banach-Mazur metric*.

The concept of a *uniform structure on the space of types* was introduced in [6] in order to give a uniform treatment of topologies on $S(T)$ such as those described above. We recall the definition here.

2.1. Definition. A *uniform structure on on the space of types* of T is a family $\mathfrak{U}$ of subsets of $S(T) \times S(T)$, called *vicinities*, such that

(1) $\mathfrak{U}$ is a base for a Hausdorff uniform structure (in the standard topological sense. See, for example, Chapter 6 of [8]) on $S(T)$.

(2) For every vicinity U there exists a set of pairs of formulas $\mathcal{D}(U)$ such that
   (i) If $(\varphi, \varphi') \in \mathcal{D}(U)$, then $\varphi < \varphi'$.
   (ii) $\mathcal{D}(U)$ *defines* U in the following sense: The pair $(p(\bar{x}), q(\bar{x}))$ is in $U \cap S(A) \times S(A)$ if and only if for every pair $(\varphi(\bar{x}, \bar{y}), \varphi'(\bar{x}, \bar{y})) \in \mathcal{D}(U)$ and every $\bar{a} \in \mathbb{Q}A$,
$$\varphi(\bar{x}, \bar{a}) \in p \quad \text{implies} \quad \varphi'(\bar{x}, \bar{a}) \in q,$$
$$\varphi(\bar{x}, \bar{a}) \in q \quad \text{implies} \quad \varphi'(\bar{x}, \bar{a}) \in p.$$
   (iii) There exists a vicinity V with the following property: If $(\varphi, \varphi') \in \mathcal{D}(U)$ and $\varphi'' > \varphi'$, there exists $(\psi, \psi') \in \mathcal{D}(V)$ such that $\varphi < \psi$ and $\psi' < \varphi''$.

2.2. Examples.

(1) *A uniform structure for the metric d.* For $\delta \geq 0$, we let
$$U_\delta = \{ (p, q) \mid d(p, q) \leq \delta \}$$
and define $\mathfrak{U}$ as the family of all sets of the form U_δ, where $\delta \in \mathbb{Q}^+$. For $\delta \in \mathbb{Q}^+$, we let $\mathcal{D}(U_\delta)$ be the set of pairs (φ, φ') such that $\varphi(\bar{x}, \bar{y})$ is of the form
$$\forall \bar{z}(\|\bar{z}\| \leq \epsilon \rightarrow \sigma(\bar{x} + \bar{z}, \bar{y})),$$
for $\epsilon > \delta$, and $\varphi'(\bar{x}, \bar{y})$ is of the form
$$\forall \bar{z}(\|\bar{z}\| \leq \epsilon - \delta \rightarrow \sigma'(\bar{x} + \bar{z}, \bar{y})),$$
with $\sigma' > \sigma$.

(2) *A uniform structure for the Banach-Mazur metric D.* For $\epsilon \geq 0$, we let
$$U_\epsilon = \{ (p, q) \mid D(p, q) \leq 1 + \epsilon \}$$
and define $\mathfrak{U}$ as the family of all sets of the form U_ϵ, where $\epsilon \in \mathbb{Q}^+$. For $\epsilon \in \mathbb{Q}^+$, we let $\mathcal{D}(U_\epsilon)$ be the set of pairs $(\varphi, \varphi_{1+\epsilon})$ such that $\varphi(\bar{x}, \bar{y})$ is an L-formula and $\varphi_{1+\epsilon}$ is the $(1 + \epsilon)$-approximation of φ introduced in [4].

For more examples of uniform structures on the space of types, we refer the reader to [6].

The following fact about uniform structures on the space of types was proved in [6].

Proposition 2.3. *Let $\mathfrak{U}$ be a uniform structure on the space of types of T. Suppose $A \subseteq B \subseteq \mathfrak{C}$. Then,*

(1) If $m \leq n$, the restriction map from $S_n(B)$ onto $S_m(A)$ is uniformly continuous with respect to $\mathfrak{U}$.

(2) If A is dense in B, the restriction map from $S_n(B)$ onto $S_n(A)$ is uniform homeomorphism with respect to $\mathfrak{U}$.

3. Stability

3.1. Definition. Let $\mathfrak{U}$ be a uniform structure on types. If κ is an infinite cardinal, a theory T is *κ-stable with respect to* $\mathfrak{U}$ if for every set $A \subseteq \mathfrak{C}$ of cardinality κ and every $n < \omega$, the density character of $S_n(A)$ with respect to the uniform topology of $\mathfrak{U}|S_n(A)$ is $\leq \kappa$.

By Proposition 2.3-(2), the word "cardinality" in the preceding definition can be replaced by "density character".

3.2. Examples. The most significant examples of stable theories known to us are (1) and (2) below; both are due to C. W. Henson.

(1) If $(X, \mathcal{B}, \mu)$ is a complete measure space, then the theory of $L_p(\mu)$ for $1 \leq p < \infty$ is ω-stable with respect to d. See [5]. (ℓ_∞ is unstable. See Example (4) below.)

(2) Let H be a Hilbert space and let $(F_i \mid i < \omega)$ be a countable family of bounded operators on H. Then the theory of the structure $(\, H, F_i \mid i < \omega \,)$ is ω-stable with respect to d. See [5].
(3) If $\mathfrak{U}$ is metrizable and κ is an infinite cardinal, then ω-stability with respect to $\mathfrak{U}$ implies κ-stability with respect to $\mathfrak{U}$.
(4) We shall see in Section 8 that every Banach space structure containing the Banach space c_0 is unstable with respect to *any* uniform structure.
(5) No theory can be ω-stable with respect to the discrete metric: the number of types over the empty set is at least $2^{\aleph_0}$.
(6) If the topology induced by $\mathfrak{U}_2$ is finer than the topology induced by $\mathfrak{U}_1$ and T is κ-stable with respect to $\mathfrak{U}_1$, then T is κ-stable with respect to $\mathfrak{U}_2$.

In rest of this is section we study stability with respect to the Banach-Mazur metric D, and compare it with stability with respect to the metric d. These results are not needed elsewhere the paper.

We assume that the language contains no real-valued relation symbols in addition to the norm, and that, except for the vector space operations, all the function symbols are constant.

3.3. Definition. Let $A \subseteq \mathfrak{C}$. We say that two tuples $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ *have the same linear dependencies over A* if there is a linear isomorphism between the span of $A \cup \{\, a_1, \dots, a_n \,\}$ and the span of $A \cup \{\, b_1, \dots, b_n \,\}$ which maps a_i to b_i, for $i = 1, \dots, n$, and fixing A pointwise.

If $p \in S_n(A)$, the linear dependencies of an n-tuple realizing p are completely determined by p, i.e., any two realizations of p have the same linear dependencies over A.

Proposition 3.4. *If $p, q \in S_n(A)$, the following conditions are equivalent.*
(1) $D(p, q) < \infty$;
(2) Any realization of p and any realization of q satisfy the same linear dependencies over A;
(3) There exist a realization of p and a realization of q which satisfy the same linear dependencies over A.

Proof. (1) $\Leftrightarrow$ (2): Let $\bar{a}$ realize p, and $\bar{b}$ realize q. Every linear isomorphism between span $A \cup \bar{a}$ and span $A \cup \bar{b}$ is a $(1+\epsilon)$–isomorphism for some $\epsilon \geq 0$, and can be extended to a $(1+\epsilon)$-automorphism of $\mathfrak{C}$.

(2) $\Leftrightarrow$ (3) follows from the remark preceding the statement of the proposition. $\dashv$

Clearly, all nonzero 1-tuples have the same linear dependencies over the empty set. Hence, by the preceding proposition, any two nonzero types over the empty set are within a finite distance of each other. However, if $A \setminus \{0\}$ is nonempty, one can find $2^{\aleph_0}$ many types over A which are infinitely apart from each other: If $a \in A \setminus \{0\}$, then, for any distinct scalars α and β, the tuples $(a, \alpha a)$ and $(a, \beta a)$ do not have the same linear dependencies. We conclude the following result.

Proposition 3.5. *No theory can be ω-stable with respect to the Banach-Mazur metric D.*

Nevertheless, it follows from Proposition 3.7 and the preceding list of examples that various important theories are κ-stable with respect to the Banach-Mazur metric, for every cardinal $\kappa \geq 2^{\aleph_0}$

The following result, relating the metrics D and d, is proved in [6]

3.6. Lemma (Perturbation Lemma for Types)**.** *Suppose that A is finite, and let p be a type over A. Then for every $\epsilon > 0$ there exists $\delta > 0$ with the following property. If*
(1) $d(p, q) < \delta$, and
(2) The realizations of p and q satisfy the same linear dependencies over A,
then $D(p, q) < 1 + \epsilon$. Furthermore, δ depends only on the quantifier-free part of p.

Proposition 3.7. *If T is κ-stable with respect to d and $\kappa \geq 2^{\aleph_0}$, then T is κ-stable with respect to the Banach-Mazur metric.*

Proof. Suppose that $\kappa \geq 2^{\aleph_0}$ and T is κ-unstable with respect to the Banach-Mazur metric. Then there exist a set A and a positive integer n of cardinality κ such that the D-density character of $S_n(A)$ is larger than κ. Therefore, there is a subset $\{ p_i \mid i < k^+ \}$ of $S_n(A)$ and $\epsilon > 0$ such that $D(p_i, p_j) > 1 + \epsilon$ for $i < j < \kappa^+$. Hence, for each pair (i, j) with $i < j < \kappa^+$ there exists a finite tuple $\bar{a}_{i,j} \in A$ such that $D(p_i|\bar{a}_{i,j}, p_j|\bar{a}_{i,j}) > \epsilon$ for $i < j < \kappa^+$ By refining (p_i) if necessary, we may assume that there exists $\bar{a} \in A$ such that $\bar{a}_{i,j} = \bar{a}$ for $i < j < \kappa^+$.

Now, there are at most $2^{\aleph_0}$ linear dependencies in n variables. Since $\kappa \geq 2^{\aleph_0}$, by further refinement of (p_i) we may assume that for $i < j < \kappa^+$ the realizations of p_i and p_j have the same linear dependencies. Fix $i < \kappa^+$. By the perturbation lemma for types, there exists $\delta_i > 0$ such that $d(\tilde{p}_i, \tilde{p}_j) > \delta_i$ for $i < j < \kappa^+$. By refining (p_i) further if necessary, we can assume that there exists $\delta > 0$ such that $\delta_i = \delta$, for $i < \kappa^+$. Thus, $d(\tilde{p}_i, \tilde{p}_j) > \delta$ for $i < j < \kappa^+$. Hence, d-density character of $S_n(\bar{a})$ is larger than κ and T is κ-unstable with respect to d. $\dashv$

Corollary 3.8. *If T is ω-stable with respect to d, then T is $2^{\aleph_0}$-stable with respect to D.*

4. Morleyizations of Types

Let $p(\bar{x})$ be a type over E, and let $\sigma(\bar{x}, y_1, \dots, y_m)$. We shall define a m-ary real-valued relation $\mathcal{P}_\sigma \colon E^m \to [0, \infty]$ such that for $\bar{a} \in E$,

$$\sigma(\bar{x}, \bar{a}) \in p \qquad \text{if and only if} \qquad \mathcal{P}_\sigma(\bar{a}) = 0.$$

The construction of $\mathcal{P}_\sigma$ is as follows. Take a set $\{ \sigma_r(\bar{x}, \bar{y}) \mid r \in \mathbb{Q}^+ \}$ of approximations of σ such that $\sigma_r < \sigma_s$ if and only if $r < s$, and $\sigma_r \to \sigma$ as $r \to 0$.

Define

$$\mathcal{P}_\sigma(\bar{a}) = \inf\{ r \in \mathbb{Q}^+ \mid \sigma_r(\bar{x}, \bar{a}) \in p \}.$$

The perturbation lemma [6] assures that $\mathcal{P}_\sigma$ is uniformly continuous on every bounded subset of E. Furthermore, it gives a modulus of uniform continuity for $\mathcal{P}_\sigma$ on each bounded subset of E. We denote by $L(\mathcal{P}_\sigma \mid \sigma \in L)$ the expansion of the language L with predicates for the real-valued relations $\mathcal{P}_\sigma$ (for $\sigma(\bar{x}, \bar{y}) \in L$) and the moduli of uniform continuity provided for them by the perturbation lemma.

We have the following chain of equivalences:

- $\sigma(\bar{x}, \bar{a}) \in p$;
- $\sigma_r(\bar{x}, \bar{a}) \in p$ for every σ_r in D;
- $\mathcal{P}_\sigma(\bar{a}) \leq r$ for every $r \in \mathbb{Q}^+$;
- $\mathcal{P}_\sigma(\bar{a}) = 0$.

We call the $L(\mathcal{P}_\sigma \mid \sigma \in L)$-structure $(E, \mathcal{P}_\sigma \mid \sigma \in L)$ the *p-morleyization* of E.

For a family of types $(p_i)_{i \in I}$, we define the $(p_i)_{i \in I}$-morleyization of E similarly: for every L-formula $\sigma(\bar{x}, \bar{y})$ with $\ell(\bar{y}) = m$ and every $i \in I$ we define a m-ary real-valued relation $\mathcal{P}_{i\sigma} \colon E^m \to [0, \infty]$ such that $\sigma(\bar{x}, \bar{a}) \in p_i$ if and only if $\mathcal{P}_{i\sigma}(\bar{a}) = 0$.

If φ and ψ are formulas with $\varphi < \psi$, $[\varphi, \psi)$ denotes the set $\{ \sigma \mid \varphi \leq \sigma < \psi \}$. The *order topology* on the language L is defined as follows. The neighborhoods of a formula φ are the sets $[\varphi, \psi)$, where $\varphi < \psi$. The following concept was introduced in [6]

4.1. Definition. A *quasi-type* over A is a set of positive bounded $L(A)$-formulas whose closure with respect to the order topology is a type.

Proposition 4.2. *Let $\Gamma(\bar{x})$ be a set of positive bounded $L(A)$-formulas such that*

(1) Γ is consistent;
(2) For some $N > 0$, the formula $\|\bar{x}\| \leq N$ is in Γ;
(3) For every positive bounded $L(A)$-formula $\varphi(\bar{x})$, either φ or $\operatorname{neg}(\varphi)$ is in Γ.

Then Γ is a quasi-type over A.

Proof. See [6]. ⊣

Proposition 4.3. *Let $p(\bar{x}) \in S(E)$. Suppose that $(F, \mathcal{P}_\sigma \mid \sigma \in L)$ is a model of $\operatorname{Th}_{\mathcal{A}}(E, \mathcal{P}_\sigma, a \mid \sigma \in L, a \in E)$, and define*

$$p^F(\bar{x}) = \{ \varphi(\bar{x}, \bar{b}) \mid \bar{b} \in F, (F, \mathcal{P}_\sigma \mid \sigma \in L) \models_{\mathcal{A}} \mathcal{P}_\varphi(\bar{b}) \leq 0 \}.$$

Then $p^F(\bar{x})$ is a quasi-type over F.

Proof. If $\varphi(\bar{x}, \bar{y})$ is an L-formula, $\psi > \varphi$, and $M > 0$,

$$(E, \mathcal{P}_\sigma \mid \sigma \in L) \not\models_{\mathcal{A}} \exists \bar{y} \Big(\|\bar{y}\| \leq M \wedge \mathcal{P}_\varphi(\bar{y}) \leq 0 \wedge \mathcal{P}_{\operatorname{neg}(\psi)}(\bar{y}) \leq 0 \Big).$$

Hence, for any pair $\varphi(\bar{x}, \bar{y}) < \psi(\bar{x}, \bar{y})$ and any $\bar{b} \in F$ with $\ell(\bar{b}) = \ell(\bar{y})$,

$$\varphi(\bar{x}, \bar{b}) \wedge \operatorname{neg}(\psi(\bar{x}, \bar{b})) \notin p^F(\bar{x}),$$

so $p^F(\bar{x})$ is consistent.

Also, for some $N > 0$, the formula $\|\bar{x}\| \leq N$ is in p. Hence,

$$(E, \mathcal{P}_\sigma \mid \sigma \in L) \models_{\mathcal{A}} \mathcal{P}_{\|\bar{x}\| \leq N} \leq 0,$$

and $\|\bar{x}\| \leq N \in p^F$.

Finally, for every formula $\varphi(\bar{x}, \bar{y})$,

$$(E, \mathcal{P}_\sigma \mid \sigma \in L) \models_{\mathcal{A}} \forall \bar{y} \Big(\|\bar{y}\| \leq M \rightarrow \mathcal{P}_\varphi(\bar{y}) \leq 0 \vee \mathcal{P}_{\operatorname{neg}(\varphi)}(\bar{y}) \leq 0 \Big);$$

hence, if $\bar{b} \in F$ and $\ell(\bar{b}) = \ell(\bar{y})$, we have $\varphi(\bar{x}, \bar{b}) \in p^F(\bar{x})$, or $\operatorname{neg}(\varphi(\bar{x}, \bar{b})) \in p^F(\bar{x})$. Thus, $p^F(\bar{x})$ is a quasi-type, by Corollary 4.2. ⊣

5. Heirs

5.1. Definition.

(1) Let $q(\bar{x})$ be a type over A, and let $\varphi(\bar{x}, \bar{y})$ be a positive bounded formula. We say that φ is *represented in* q if there exists $\bar{a} \in A$ such that $\varphi(\bar{x}, \bar{a}) \in q$. We say that φ is *almost represented in* q if every approximation of φ is represented in q.
(2) If $A \supseteq E$, $p(\bar{x})$ is a type over E and $q(\bar{x})$ is an extension of p over A, we say that q is an *heir* of p if every $L(E)$-formula that is represented in q is almost represented in p.

Proposition 5.2. *Let $p(\bar{x}) \in S(E)$. Suppose that $(F, \mathcal{P}_\sigma \mid \sigma \in L)$ is a model of $\operatorname{Th}_{\mathcal{A}}(E, \mathcal{P}_\sigma, a \mid \sigma \in L, a \in E)$, and let p^F be as in Proposition 4.3. Then the closure of $p^F(\bar{x})$ is an heir of $p(\bar{x})$.*

Proof. We proved in Proposition 4.3 that the closure of p^F is a type. If $\varphi(\bar{x}, \bar{y})$ is an $L(E)$-formula that is represented in the closure of $p^F(\bar{x})$, then

$$(F, \mathcal{P}_\sigma \mid \sigma \in L) \models_{\mathcal{A}} \exists \bar{y}(\|\bar{y}\| \leq N \wedge \mathcal{P}_\varphi(\bar{y}) \leq 0)$$

for some $N > 0$. But then,

$$(E, \mathcal{P}_\sigma \mid \sigma \in L) \models_{\mathcal{A}} \exists \bar{y}(\|\bar{y}\| \leq N \wedge \mathcal{P}_\varphi(\bar{y}) \leq 0),$$

and hence φ is almost represented in $p(\bar{x})$. ⊣

5.3. Lemma. *Suppose $E \prec_{\mathcal{A}} F$. Let $p \in S(E)$ and let q_1, q_2 be heirs of p over F. If $E' \succ_{\mathcal{A}} E$ and p' is an heir of p over E', there exist $E'' \succ_{\mathcal{A}} E'$, heirs r_1, r_2 of p' over E'', and an embedding $f\colon F \prec_{\mathcal{A}} E''$ such that $f(q_1) \subseteq r_1$ and $f(q_2) \subseteq r_2$.*

Proof. Let b' denote a new constant for each $b \in E' \setminus E$. Let

$$(\, F, \mathcal{Q}_{1\varphi}, \mathcal{Q}_{2\varphi} \mid \varphi \in L\,)$$

be the (q_1, q_2)-morleyization of F. Every $L(E)$-formula which is represented in p' is almost represented in p, and hence almost represented in q_1 and q_2. Therefore, the theory

$$\begin{aligned} \Sigma = \quad & \mathrm{Th}_{\mathcal{A}}(F, \mathcal{Q}_{1\sigma}, \mathcal{Q}_{2\sigma}, b, \mid \sigma \in L, b \in F) \\ & \cup\{\, \mathcal{Q}_{1\tau}(\bar{a}, \bar{b}') \leq 0 \mid \bar{a} \in E,\ \bar{b} \in E' \setminus E,\ \sigma(\bar{x}, \bar{a}, \bar{b}) \in p' \text{ for some } \sigma < \tau \,\} \\ & \cup\{\, \mathcal{Q}_{2\tau}(\bar{a}, \bar{b}') \leq 0 \mid \bar{a} \in E,\ \bar{b} \in E' \setminus E,\ \sigma(\bar{x}, \bar{a}, \bar{b}) \in p' \text{ for some } \sigma < \tau \,\} \end{aligned}$$

is finitely consistent.

Let $\tilde{F}$ be the reduct of a model of Σ to the language L. Then the following conditions hold:

(1) $\tilde{F} \succ_{\mathcal{A}} F$;
(2) There exists an extension $\tilde{q}_1$ of q_1 over F such that every $L(E)$-formula which represented in $\tilde{q}_1$ is almost represented in q_1;
(3) There exists an extension $\tilde{q}_2$ of q_2 over F such that every $L(E)$-formula which represented in $\tilde{q}_2$ is almost represented in q_2;
(4) There exists an embedding $g\colon E' \prec_{\mathcal{A}} \tilde{F}$ fixing E pointwise, such that $g(p') \subseteq \tilde{q}_1, \tilde{q}_2$.

Since $\mathrm{tp}(E'/E) = \mathrm{tp}(g(E')/E)$, there exists an automorphism h of the monster model such that $h = g^{-1}$ on $g(E')$. Define $f = h|F$, $E'' = f(\tilde{F})$, $r_1 = f(\tilde{q}_1)$, and $r_2 = f(\tilde{q}_2)$. ⊣

6. Definable Types

Recall that If B is a subset of the monster model, $\mathcal{B}(B)$ denotes the set of elements of B of norm less than or equal to 1.

6.1. Definition. Let $A \subseteq B$. A type $p(\bar{x}) \in S(B)$ is called *definable over A* if for every pair of L-formulas $\varphi(\bar{x}, \bar{y}) < \psi(\bar{x}, \bar{y})$ there exists a pair of $L(A)$-formulas

$$d_1^{\varphi,\psi}(\bar{y}) < d_2^{\varphi,\psi}(\bar{y})$$

such that for all $\bar{a} \in \mathcal{B}(\mathbb{Q}B)$,

$$\begin{aligned} \varphi(\bar{x}, \bar{a}) \in p(\bar{x}) \quad & \text{implies} \quad \models d_1^{\varphi,\psi}(\bar{a}) \\ \models d_2^{\varphi,\psi}(\bar{a}) \quad & \text{implies} \quad \psi(\bar{x}, \bar{a}) \in p. \end{aligned}$$

The map $(\varphi, \psi) \mapsto (d_1^{\varphi,\psi}, d_2^{\varphi,\psi})$ is called a *definition schema* for p over A. A type $p \in S(B)$ is *definable* if it is definable over B.

If $p \in S(E)$ and q is an extension of p, any definition schema for q is a definition schema for p. A definition schema for p need not be a definition schema for q; however, Lemma 6.3 shows that if q is an heir of p, any sufficiently close "approximation" of a definition schema for p will be a definition schema for q.

6.2. Lemma. *Let $A \subseteq B$ and $p(\bar{x}) \in S(B)$. Let $\epsilon > 0$, and suppose that for every pair of L-formulas $\varphi(\bar{x}, \bar{y}) < \psi(\bar{x}, \bar{y})$ there exists a pair of $L(A)$-formulas*

$$\theta_1^{\varphi,\psi}(\bar{y}) < \theta_2^{\varphi,\psi}(\bar{y})$$

such that for $\bar{a} \in \mathcal{B}_{1-\epsilon}(\mathbb{Q}B)$,

$$\varphi(\bar{x}, \bar{a}) \in p(\bar{x}) \qquad \textit{implies} \qquad \models_{\mathcal{A}} \theta_1^{\varphi,\psi}(\bar{a})$$

$$\models_{\mathcal{A}} \theta_2^{\varphi,\psi}(\bar{a}) \qquad \textit{implies} \qquad \psi(\bar{x}, \bar{a}) \in p.$$

Then p is definable over A. Furthermore, if

$$\tilde{\varphi}(\bar{x}, \bar{y}) = \varphi(\bar{x}, (1-\epsilon)\bar{y}), \qquad \tilde{\psi}(\bar{x}, \bar{y}) = \psi(\bar{x}, (1-\epsilon)\bar{y}),$$

then

$$(\, \varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}) \,) \mapsto (\, d_1^{\tilde{\varphi},\tilde{\psi}}(\bar{y}), d_2^{\tilde{\varphi},\tilde{\psi}}(\bar{y}) \,)$$

is a definition schema for p.

The proof of Lemma 6.2 is a simple exercise.

6.3. Lemma. *Suppose that $E \prec_{\mathcal{A}} F$, $p(\bar{x}) \in S(E)$ and q is an heir of p over F. Suppose also that*

$$(\, \varphi(\bar{x}, \bar{y}),\ \psi(\bar{x}, \bar{y}) \,) \mapsto (\, d_1^{\psi}\varphi(\bar{y}),\ d_2^{\psi}\varphi(\bar{y}) \,)$$

is a definition schema for p. Let $\varphi'(\bar{x}, \bar{y})$ and $\theta^{\varphi,\psi}(\bar{x}, \bar{y})$ be formulas such that
(1) $\varphi(\bar{x}, \bar{y}) < \varphi'(\bar{x}, \bar{y}) < \psi(\bar{x}, \bar{y})$;
(2) $d_2^{\varphi',\psi}(\bar{y}) < \theta^{\varphi,\psi}(\bar{y})$.
Then, if $\epsilon > 0$ and

$$\tilde{\varphi}(\bar{x}, \bar{y}) = \varphi(\bar{x}, (1-\epsilon)\bar{y}), \qquad \tilde{\psi}(\bar{x}, \bar{y}) = \psi(\bar{x}, (1-\epsilon)\bar{y}),$$

the map

$$(\, \varphi(\bar{x}, \bar{y}),\ \psi(\bar{x}, \bar{y}) \,) \mapsto (\, d_1^{\tilde{\varphi}',\tilde{\psi}}(\bar{y}),\ \theta^{\tilde{\varphi}',\tilde{\psi}}(\bar{y}) \,)$$

is a definition schema for q.

Proof. Fix L-formulas $\varphi(\bar{x}, \bar{y}) < \psi(\bar{x}, \bar{y})$. Choose formulas φ' and θ such that $\varphi < \varphi' < \psi$ and $d_2^{\varphi',\psi} < \theta$.

Step 1. *If $\|\bar{a}\| \leq 1-\epsilon$ and $\varphi(\bar{x}, \bar{a}) \in q$, then $\models d_1^{\varphi',\psi}(\bar{a})$.*

Proof of Step 1. Take $d_1^{\varphi',\psi} < \gamma < \gamma'$. It suffices to prove that the formula

(*) $$\|\bar{y}\| \leq 1-\epsilon \wedge \varphi(\bar{x}, \bar{y}) \wedge \operatorname{neg}(\gamma'(\bar{y}))$$

is not represented in q. If it were represented in q, it would be almost represented in p (since q is an heir of p), and since

$$\|\bar{y}\| \leq 1 \wedge \varphi'(\bar{x}, \bar{y}) \wedge \operatorname{neg}(\gamma(\bar{y}))$$

is an approximation of $(*)$, there would be $\bar{a} \in E$ such that

$$\|\bar{a}\| \leq 1 \wedge \varphi'(\bar{x}, \bar{a}) \wedge \operatorname{neg}(\gamma(\bar{a})) \in p(\bar{x}).$$

Since $\gamma > d_1^{\varphi',\psi}$, this contradicts the fact that $\varphi'(\bar{x},\bar{a}) \in p$ implies $\models_{\mathcal{A}} d_1^{\varphi',\psi}$. ⊣

Step 2. *If $\|\bar{a}\| \leq 1-\epsilon$ and $\models \theta_2(\bar{a})$, then $\psi(\bar{x},\bar{a}) \in q$*

Proof of Step 2. Take $\psi < \psi' < \psi''$. It suffices to prove that the formula

$$\|\bar{y}\| \leq 1-\epsilon \wedge \theta(\bar{y}) \wedge \mathrm{neg}(\psi''(\bar{x},\bar{y}))$$

is not represented in q. If it were, it would be almost represented in p, and there would be $\bar{a} \in E$ such that

$$\|\bar{a}\| \leq 1 \wedge d_2^{\varphi',\psi}(\bar{a}) \wedge \mathrm{neg}(\psi'(\bar{x},\bar{a})) \in p(\bar{x}).$$

Since $\psi' > \psi$, this contradicts the fact that $\models_{\mathcal{A}} d_2^{\varphi',\psi}(\bar{a})$ implies $\psi(\bar{x},\bar{a}) \in p$. ⊣

The lemma follows from Steps 1 and 2, and Lemma 6.2. ⊣

6.4. Theorem. *Let $\mathfrak{U}$ be a uniform structure on the space of types. The following conditions are equivalent for any $p(\bar{x}) \in S(E)$:*

(1) p is definable.
(2) For each $F \succ_{\mathcal{A}} E$, p has a unique heir over F.
(3) There exists a cardinal κ such that whenever $F \succ_{\mathcal{A}} E$ and density$(F) \leq \kappa$, *the set of heirs of p over F has density character $\leq \kappa$ with respect to the uniform topology of $\mathfrak{U}|S(F)$.*

Proof. (1) $\Rightarrow$ (2) : Let q be an heir of p over F. Lemma 6.3 exhibits a definition schema for q, given one for p. This determines q.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2): Let $\mathfrak{U}$ be a uniform structure on types. Suppose, by way of contradiction, that there exist a model $F \succ_{\mathcal{A}} E$, heirs q_1, q_2 of p over F, and a vicinity $U \in \mathfrak{U}$ such that $(q_1,q_2) \notin U$. Let κ be a cardinal with $\kappa >$ density(E) and let γ be the least cardinal such that $2^\gamma > \kappa$. Using Lemma 5.3 iteratively, one can construct a model $E' \succ_{\mathcal{A}} E$ of density character $\leq \kappa$, and a set $\{ p_i \mid i < \gamma \}$ of heirs of p over E' such that $(p_i, p_j) \notin U$, for $i < j < \gamma$.

(2) $\Rightarrow$ (1): Let $(E, \mathcal{P}_\sigma \mid \sigma \in L)$ be the p-morleyization of E and let $\Sigma(P_\sigma \mid \sigma \in L)$ be its complete theory. We have seen that if

$$(F, \mathcal{P}_\sigma^F \mid \sigma \in L) \models_{\mathcal{A}} \Sigma(\mathcal{P}_\sigma \mid \sigma \in L),$$

then p has an heir over F, namely, p^F. The fact that p has a unique heir over any $F \succ_{\mathcal{A}} E$ means that $\Sigma(\mathcal{P}_\sigma \mid \sigma \in L)$ defines $\{ \mathcal{P}_\sigma \mid \sigma \in L \}$ implicitly. But then, by the Beth Definability Theorem [6], $\Sigma(\mathcal{P}_\sigma \mid \sigma \in L)$ defines $\{ \mathcal{P}_\sigma \mid \sigma \in L \}$ explicitly. Thus, for every L-formula $\sigma(\bar{x},\bar{y})$ and every rational number $r > 0$ there exists a L-formula $\theta(\bar{y})$ such that

$$\mathcal{P}_\sigma(\bar{y}) \leq 0 \wedge \|\bar{y}\| \leq 1 \models_{\mathcal{A}} \theta(\bar{y}) \tag{*}$$

Fix $r > s$. By the compactness theorem there exists $\theta' > \theta$ such that

$$\theta'(\bar{y}) \wedge \|\bar{y}\| \leq 1 \models_{\mathcal{A}} \mathcal{P}_\sigma(\bar{y}) \leq s. \tag{**}$$

Since s can be taken arbitrarily small, $(*)$ and $(**)$ prove that p is definable. ⊣

7. Stability and Definability of Types

A theory is *stable* with respect to $\mathfrak{U}$ if there exists an infinite cardinal κ such that T is κ-stable with respect to $\mathfrak{U}$. We shall prove that this condition is independent of the particular uniform structure $\mathfrak{U}$. (Corollary 7.3.)

7.1. Lemma. *Suppose that $E \prec_{\mathcal{A}} F$, $p \in S(E)$ and q is an extension of p over F. If A is a dense subset of E and every $L(A)$-formula that is represented in q is almost represented in p, then q is an heir of p.*

Proof. Let $(E, \mathcal{P}_\sigma \mid \sigma \in L)$ be the p-morleyization of E, and let $(F, \mathcal{Q}_\sigma \mid \sigma \in L)$ be the q-morleyization of F. Since every $L(A)$-formula which is almost represented in q is also almost represented in p,

$$(F,\ \mathcal{Q}_\sigma,\ a \mid \sigma \in L\ , a \in A) \equiv_{\mathcal{A}} (E,\ \mathcal{P}_\sigma,\ a \mid \sigma \in L\ , a \in A).$$

Since A is dense in E, the perturbation lemma implies that

$$(F,\ \mathcal{Q}_\sigma,\ a \mid \sigma \in L, a \in E) \equiv_{\mathcal{A}} (E,\ \mathcal{P}_\sigma,\ a \mid \sigma \in L,\ a \in E).$$

But then every $L(E)$-formula that is almost represented in q is represented in p. Therefore, every $L(E)$-formula which is represented in q is almost represented in p, i.e., q is an heir of p. $\dashv$

Corollary 7.2. *Let $\mathfrak{U}$ be a uniform structure on types. The following conditions are equivalent:*

(1) T is stable with respect to $\mathfrak{U}$.

(2) Every type over a model is definable.

Proof. (1) $\Rightarrow$ (2): Suppose that $p \in S(E)$ is undefinable. Take a separable $E_0 \prec F$ such that p is an heir of $p|E_0$. Lemma 6.3 implies that $p|E_0$ is undefinable. Since E_0 is separable, (3) $\Rightarrow$ (1) of Theorem 6.4 implies that for every cardinal κ there exists $F \succ_{\mathcal{A}} E_0$ such that density$(F) \leq \kappa$ and the set of heirs of $p|E_0$ over F has density character $> \kappa$ with respect to $\mathfrak{U}|S(F)$. Therefore, T is not stable with respect to $\mathfrak{U}$.

(2) $\Rightarrow$ (1): If κ be a cardinal such that $\kappa^{\aleph_0} = \kappa$ and E be a model of T of density character κ, there are at most κ many types in $S(E)$, since the number of definition schemata for types over E cannot exceed the number of definition schemata, i.e., $(\kappa^{\aleph_0})^{\aleph_0} = \kappa$. $\dashv$

The condition of a type's being definable does not make reference to any uniform structure. Hence, we can conclude the following:

Corollary 7.3. *If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are uniform structures on types, T is stable with respect to $\mathfrak{U}_1$ if and only if T is stable with respect to $\mathfrak{U}_2$.*

Remark. Since T's being stable with respect to some uniform structure $\mathfrak{U}$ is independent of $\mathfrak{U}$, hereafter we shall express this fact simply as "T is stable".

8. Stability and Order

8.1. Definition. Let E be a model. A positive formula $\theta(\bar{x}, \bar{y})$ with $\ell(\bar{x}) = \ell(\bar{y})$ *defines an order* in E if there exist a bounded sequence $(\bar{a}_k)_{k<\omega}$ in E and a formula $\theta' > \theta$ such that

$$E \models_{\mathcal{A}} \theta(\bar{a}_k, \bar{a}_l), \qquad \text{if } k \leq l;$$

$$E \models_{\mathcal{A}} \operatorname{neg}(\theta'(\bar{a}_k, \bar{a}_l)), \qquad \text{if } k > l.$$

In this case, we say that the pair θ, θ' *orders* the sequence $(\bar{a}_k)_{k<\omega}$.

Example 8.2. Let $(e_k)_{k<\omega}$ be the standard basis for the Banach space c_o, and let $a_k = e_0 + \cdots + e_k$. Then

$$\|e_k + a_l\| = \begin{cases} 2, & \text{if } k \le l \\ 1, & \text{if } k > l. \end{cases}$$

Thus the pair of formulas

$$\theta(x, y; u, v)\colon \|y + u\| \ge 2, \qquad \theta'(x, y; u, v)\colon \|y + u\| \ge 1$$

orders the sequence $(a_k \frown e_k)_{k<\omega}$ in c_0.

Remarks. (1) Let E be a Banach space, E^* be its dual, and define $F = E \oplus_\infty E^*$. Let $\mathcal{R}$ be the pairing function $(x, \xi) \mapsto < x, \xi >$ on F. Then, if $r < s$, the pair of formulas $\mathcal{R}(\bar{x}, \bar{y}) \le r$, $\mathcal{R}(\bar{x}, \bar{y}) \le s$ orders a sequence in F if and only if E is not reflexive. This is the *James Condition* for reflexivity, introduced by R. C. James [7]. (See also Section 3.5 of [1].)

(2) If the sequence $(\bar{a}_k)_{k<\omega}$ is ordered by a pair of formulas, then there exists $\delta > 0$ such that $\|a_i - a_j\| \ge \delta$ for $i < j < \omega$. (*Proof:* Suppose that no such δ exists. Then $(\bar{a}_k)_{k<\omega}$ has a Cauchy subsequence $(\bar{a}_{k_i})_{i<\omega}$. Let $a = \lim_{i\to\infty} \bar{a}_{k_i}$. Assume that θ, θ' is a pair of formulas ordering $(\bar{a}_k)_{k<\omega}$. By the perturbation lemma, we must have $E \models_{\mathcal{A}} \theta(\bar{a}, \bar{a})$ and $E \models_{\mathcal{A}} \text{neg}(\theta'(\bar{a}, \bar{a}))$, which is, of course, impossible.)

(3) By (2), no formula can define an order in a finite-dimensional space.

(4) Suppose that the pair $\theta(\bar{x}, \bar{y}), \theta'(\bar{x}, \bar{y})$ orders the sequence $(\bar{a}_k)_{k<\omega}$. Then θ, θ' *strictly* orders $(\bar{a}_k)_{k<\omega}$, i.e.,

$$E \models_{\mathcal{A}} \varphi(\bar{a}_k, \bar{a}_l), \qquad \text{if } k < l;$$
$$E \models_{\mathcal{A}} \text{neg}(\varphi'(\bar{a}_k, \bar{a}_l)), \qquad \text{if } k > l.$$

Suppose, conversely, that there exists a pair φ, φ' that strictly orders the sequence $(\bar{a}_k)_{k<\omega}$ as above. Let δ be as in (2), and take $\epsilon < \delta$. Then the pair of formulas

$$\theta(\bar{x}, \bar{y})\colon \varphi(\bar{x}, \bar{y}) \vee \|\bar{x} - \bar{y}\| \le \epsilon,$$
$$\theta'(\bar{x}, \bar{y})\colon \varphi'(\bar{x}, \bar{y}) \vee \|\bar{x} - \bar{y}\| \le \delta$$

orders $(\bar{a}_k)_{k<\omega}$.

Thus, a sequence is ordered by a pair of formulas if and only if it is strictly ordered by another pair of formulas.

8.3. Definition. Let $E \subseteq A$ and $q(\bar{x}) \in S(A)$. We say that q is *finitely realized in* E if for every positive bounded formula $\varphi(\bar{x}, \bar{a}) \in q$ and every $\psi > \varphi$ there exists $\bar{u} \in E$ such that $E \models_{\mathcal{A}} \psi(\bar{u}, \bar{a})$.

8.4. Lemma. *Assume that no formula defines an order in E. Then, if $p(\bar{x}) \in S(E)$, every heir of p is finitely realized in E.*

Proof. Suppose that there exist $A \supseteq E$ and an heir $q(\bar{x})$ of $p(\bar{x})$ over A which is not finitely realized in E. Then there exist positive bounded formulas $\varphi(\bar{a}, \bar{x}) \in q(\bar{x})$ and $\psi > \varphi$ such that

(*) $$\text{for any } \bar{u} \in E, \qquad \not\models_{\mathcal{A}} \psi(\bar{a}, \bar{u}).$$

Let $\bar{b}$ be a realization of $q(\bar{x})$. Then

(**) $$\models_{\mathcal{A}} \varphi(\bar{a}, \bar{b}).$$

Fix rational numbers $N > M \ge \|\bar{a}\|, \|\bar{b}\|$. Fix also formulas $\varphi < \varphi_0 < \varphi_1 < \cdots < \varphi' < \psi$. We construct, by induction on k, sequences $(\bar{a}_k)_{k<\omega}$ and $(\bar{b}_k)_{k<\omega}$ in E such that

(1) $\|\bar{a}_k\|, \|\bar{b}_k\| \leq N$;
(2) $\models_{\mathcal{A}} \varphi_k(\bar{a}_k, \bar{b})$, for all k;
(3) $\models_{\mathcal{A}} \varphi_k(\bar{a}_k, \bar{b}_l)$, if $k \leq l$;
(4) $\models_{\mathcal{A}} \operatorname{neg}(\psi(\bar{a}_k, \bar{b}_k))$, if $k > l$.

But then we will be done, because then, by (3) and (4), the pair of formulas

$$\theta(\bar{x} \frown \bar{y}, \bar{x} \frown \bar{y}) \colon \varphi'(\bar{x}, \bar{y}) \qquad \theta'(\bar{x} \frown \bar{y}, \bar{x} * \bar{y}) \colon \psi(\bar{x}, \bar{y})$$

orders the sequence $(a_k \frown b_k)_{k<\omega}$. (Condition (2) is used only to carry out the induction.)

Assume that (1)–(4) hold for a positive integer k, in order to prove the corresponding statements for $k+1$.

By $(*)$ there exists a positive bounded formula $\rho > \psi$ such that

$$\models_{\mathcal{A}} \bigwedge_{i \leq k} \operatorname{neg}(\rho(\bar{a}, \bar{b}_i)).$$

By (2) of the induction hypothesis and $(**)$,

$$\models_{\mathcal{A}} \|\bar{a}\| \leq M \wedge \|\bar{b}\| \leq M \wedge \varphi(\bar{a}, \bar{b}) \wedge \bigwedge_{i \leq k} \varphi_i(\bar{a}_i, \bar{b}) \wedge \bigwedge_{i \leq k} \operatorname{neg}(\rho(\bar{a}, \bar{b}_i)).$$

Now take $\psi < \psi' < \lambda$. Since $q = \operatorname{tp}(\bar{b}/A)$ is an heir of $p = \operatorname{tp}(\bar{b}/E)$, there exists $\bar{a}' \in E$ such that

$$\models_{\mathcal{A}} \operatorname{neg}(\|\bar{a}'\| \leq N \wedge \|\bar{b}\| \leq M \wedge \varphi_0(\bar{a}', \bar{b}) \wedge \bigwedge_{i \leq k} \varphi_i(\bar{a}_i, \bar{b}) \wedge \bigwedge_{i \leq n} \operatorname{neg}(\psi'(\bar{a}', \bar{b}_i)),$$

and since E is a model, there exists $\bar{b}' \in E$ such that

$$\models_{\mathcal{A}} \|\bar{a}'\| \leq N \wedge \|\bar{b}'\| \leq N \wedge \varphi_{i+1}(\bar{a}', \bar{b}') \wedge \bigwedge_{i \leq k} \varphi_{i+1}(\bar{a}_i, \bar{b}') \wedge \bigwedge_{i \leq k} \operatorname{neg}(\psi'(\bar{a}', \bar{b}_i)).$$

Thus, (1)–(4) are satisfied with $\bar{a}_{k+1} = \bar{a}'$ and $\bar{b}_{k+1} = \bar{b}'$. ⊣

8.5. Lemma. *The cardinality of set of types that are finitely realized in a model E is bounded by $2^{2^{\operatorname{card}(E)}}$.*

Proof. If $\theta(x_1, \ldots, x_n, \bar{a})$ is a positive bounded formula and E is a model, we denote by $\theta(E, \bar{a})$ the subset of E^n defined by $\theta(\bar{x}, \bar{a})$, i.e.,

$$\theta(E, \bar{a}) = \{ (b_1, \ldots, b_n) \in E \ \mid \models_{\mathcal{A}} \theta(b_1, \ldots, b_n, \bar{a}) \}.$$

For each n-type $q(\bar{x})$ that is finitely realized in E, let

$$E(q) = \{ \varphi'(E, \bar{a}) \mid \varphi(\bar{x}, \bar{a}) \in q(\bar{x}) \text{ and } \varphi < \varphi' \}.$$

Then (1) $E(q) \subseteq \mathcal{P}(E^n)$, (2) $\emptyset \notin E(q)$, and (3) $E(q)$ is closed under intersections. We prove the lemma by showing that if $q(\bar{x})$ and $r(\bar{x})$ are distinct n-types that are finitely realized in E, then $E(q) \neq E(r)$.

Suppose that $q(\bar{x})$ and $r(\bar{x})$ are distinct. Then there exist $\varphi(\bar{x}, \bar{a})$ and $\psi > \varphi$ such that $\varphi(\bar{x}, \bar{a}) \in q$ and $\operatorname{neg}(\psi(\bar{x}, \bar{a})) \in r$. Take formulas $\varphi < \varphi' < \varphi'' < \psi$. Then $\varphi'(E, \bar{a}) \in E(q)$ and $\operatorname{neg}(\varphi''(E, \bar{a})) \in E(r)$. But then, $E(r) \neq E(q)$: otherwise, by (3), we would have $\emptyset = \varphi'(E, \bar{a}) \cap \operatorname{neg}(\varphi''(E, \bar{a})) \in E(q)$, which is in contradiction with (2). ⊣

8.6. Definition. Let $\varphi(\bar{x},\bar{y})$, $\psi(\bar{x},\bar{y})$ be positive bounded formulas and let $\varphi' > \varphi$. Let $M \geq 0$. We define $R[\psi,\varphi,\varphi',M]$ to be the largest positive integer N with the following property: there exist a model E and sequences $(\bar{a}_s)_{s\in 2^N}$ $(\bar{b}_t)_{t\in 2^{<N}}$ in $\mathcal{B}_M$ such that:

$$\begin{aligned} &E \models \psi(\bar{a}_s,\bar{b}_t), && \text{for } s \in 2^N,\, t \in 2^{<N};\\ &E \models \varphi(\bar{a}_s,\bar{b}_{s|i}), && \text{if } s(i)=0;\\ &E \models \operatorname{neg}(\varphi'(\bar{a}_s,\bar{b}_{s|i})), && \text{if } s(i)=1. \end{aligned}$$

If such a largest N does not exist, we write $R[\psi,\varphi,\varphi',M]=\infty$.

8.7. Theorem. *The following conditions are equivalent:*

(1) T is stable.
(2) For any $\psi(\bar{x},\bar{y})$, any $\varphi(\bar{x},\bar{y})$ and $\varphi' > \varphi$, $R[\psi,\varphi,\varphi',M]$ is finite.
(3) No formula can define an order in a model of T.
(4) If $p \in S(E)$, every heir of p is finitely realized in E.

Proof. (1) $\Rightarrow$ (2): Suppose that $R[\psi,\varphi,\varphi',M]$ is infinite. Then, by the compactness theorem, for any infinite cardinal κ, the set

$$\begin{aligned} \Sigma_\kappa = \;& \{\, \|\bar{x}_s\| \leq M \wedge \|\bar{y}_t\| \leq M \,\}\\ & \cup\{\, \psi(\bar{x}_s,\bar{y}_t) \mid s\in 2^\kappa, t \in 2^{<\kappa}\,\}\\ & \cup\{\, \varphi(\bar{x}_s,\bar{y}_{s|i}) \mid s\in 2^\kappa, i<\kappa, s(i)=0\,\}\\ & \cup\{\operatorname{neg}(\varphi'(\bar{x}_s,\bar{y}_{s|i})) \mid s\in 2^\kappa, i<\kappa, s(i)=1\,\} \end{aligned}$$

is consistent. Fix an infinite cardinal λ, in order to prove that T is λ-unstable. Take κ minimal with the property $2^\kappa > \lambda$. Suppose that $(\bar{b}_s)_{s\in 2^\kappa}$ and $(\bar{a}_t)_{t\in 2^{<\kappa}}$ realize Σ_κ, and let $A = \bigcup\{\, a_t \mid t \in 2^{<\kappa}\,\}$. Then $\operatorname{card}(A) \leq \lambda$ and $\operatorname{tp}(c_s/A) \neq \operatorname{tp}(c_{s'}/A)$ for $s \neq s'$ in 2^κ. Thus T is not stable with respect to the discrete uniform structure.

(2) $\Rightarrow$ (3): Suppose that the pair θ,θ' orders the bounded sequence $(\bar{a}_k)_{k<\omega}$ in E. Let $M > 0$ be a bound for this sequence. It is easy to see that $\bar{a}_0,\dots,a_{2^k}$ witness the fact that $R[\|\bar{x}\| \leq M, \theta(\bar{x},\bar{y}), \theta'(\bar{x},\bar{y}), M] \geq k$.

(3) $\Rightarrow$ (4) is Lemma 8.4.

(4) $\Rightarrow$ (1): Let $p \in S(E)$. Lemma 8.5 gives a uniform bound for the number of extensions of p (over any $A \supseteq E$) that are finitely realized in E. Hence, under the assumption (4), the number of heirs of p over any extension of E is uniformly bounded. But then p is definable by Theorem 6.4. Thus, T is stable by Corollary 7.2. $\dashv$

Remark. Stability is not preserved under distortion of the norm, even by small amounts. For $\epsilon > 0$, define a new norm $\|\;\|_\epsilon$ on ℓ_2 as follows. For $x=(x_i)_{i\in\omega}$ in ℓ_2, let

$$\|x\|_\epsilon = \|x\| + \epsilon\, \sup\{\, |x_{2i}+x_{2j-1}| \mid i<j<\omega\,\}$$

Then, if e_k is the k-th unit vector, we have

$$\|e_{2k}+e_{2l-1}\|_\epsilon = \begin{cases} \sqrt{2}+2\epsilon, & \text{if } k<l\\ \sqrt{2}+\epsilon, & \text{if } k>l. \end{cases}$$

Thus, ℓ_2 with the norm $\|\;\|_\epsilon$ is not stable, by Theorem 8.7. (This example was taken from [2], where the credit is given to B. Bollobás. A similar example is found in [13].)

9. Stability and Iterated Limits

Let E be a Banach space structure and let $N > 0$. Recall from [6] that an m-ary *real-valued relation* on $\mathcal{B}_N(E)$ is a function $\mathcal{R}\colon \mathcal{B}_N^m(E) \to [-\infty, \infty]$ which is uniformly continuous on $\mathcal{B}_N(E)$.

The concept of *definable real-valued relation* was also introduced in [6]:

An m-ary real-valued relation $\mathcal{R}$ on $\mathcal{B}_N(E)$ is *definable* if for every pair of rational numbers $r < s$ there exists a pair of positive bounded $\theta_1^{r,s}(x_1 \dots x_m) < \theta_2^{r,s}(x_1 \dots x_m)$ such that for every $\bar{a} \in \mathcal{B}_N(E)$,

$$\begin{array}{ccc} \mathcal{R}(\bar{a}) \le r & \text{implies} & \models \theta_1^{r,s}(\bar{a}), \\ \models \theta_2^{r,s}(\bar{a}) & \text{implies} & \mathcal{R}(\bar{a}) \le s. \end{array}$$

Let E be a Banach space structure and let $\mathcal{R}$ be a real-valued relation on $\mathcal{B}_N(E)$. A bounded sequence $(b_n)_{n<\omega}$ in E is called *$\mathcal{R}$-approximating* if the limit $\lim_{n\to\infty} \mathcal{R}(\bar{b}_n, \bar{x})$ exists for every $x \in \mathcal{B}_N(E)$. If E is a separable structure and $\mathcal{R}$ is a real-valued relation on $\mathcal{B}_N(E)$, then every bounded sequence in $\mathcal{B}_N(E)$ has an $\mathcal{R}$-approximating subsequence.

9.1. Theorem. *The following conditions are equivalent.*

(1) T stable.

(2) If E is a separable model of T, $\mathcal{R}(\bar{x}, \bar{y})$ is a definable real-valued relation on $\mathcal{B}_N(E)$, and $(\bar{b}_m)$ and $(\bar{c}_n)$ are approximating sequences in $\mathcal{B}_N(E)$, then the double limits

$$\lim_m \bigl(\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr), \qquad \lim_n \bigl(\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr)$$

exist and are equal.

(3) If E is a model of T, $\mathcal{R}(\bar{x}, \bar{y})$ is a definable real-valued relation $\mathcal{B}_N(E)$, and $(\bar{b}_m)$ and $(\bar{c}_n)$ are approximating sequences in $\mathcal{B}_N(E)$, then

$$\lim_m \bigl(\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr) = \lim_n \bigl(\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr),$$

where the equality means that if the limits exist, they are equal.

Proof. The equivalence between (2) and (3) is clear. We prove the equivalence between (1) and (2).

(1) $\Rightarrow$ (2): Suppose that condition (2) fails for a real-valued relation $\mathcal{R}(\bar{x}, \bar{y})$ on $\mathcal{B}_N(E)$ approximating sequences $(\bar{b}_m)$ and $(\bar{c}_n)$. Then, either

$$\liminf_m \bigl(\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr) < \limsup_n \bigl(\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr),$$

or

$$\liminf_n \bigl(\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr) < \limsup_m \bigl(\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr).$$

Assume the former case (the latter is, of course, symmetric). By extracting subsequences, we may assume that both limits

$$\lim_m \bigl(\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr), \qquad \lim_n \bigl(\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr)$$

exist. Then

$$(*) \qquad l = \lim_m \bigl(\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr) < \lim_n \bigl(\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n)\bigr) = L.$$

Take rational numbers r and s such that $l < r < s < L$. We will use $(**)$ to find subsequences $(\bar{b}_{n_i})$ and $(\bar{c}_{n_j})$ of $(\bar{c}_n)$ such that

$$\mathcal{R}(\bar{b}_{n_i}, \bar{c}_{n_j}) < r, \qquad \text{for } i < j;$$
$$\mathcal{R}(\bar{b}_{n_i}, \bar{c}_{n_j}) > s, \qquad \text{for } i > j.$$

Since $\mathcal{R}$ is definable, this will prove that T has an order and is is therefore unstable by Theorem 8.7. By $(*)$, there exist m_1 and n_1 such that

(i) $\lim_n \mathcal{R}(\bar{b}_m, \bar{c}_n) < r$, for $m \geq m_1$;
(ii) $\lim_m \mathcal{R}(\bar{b}_m, \bar{c}_n) > s$, for $n \geq n_1$.

Suppose that $m_1 < \cdots < m_k$ and $n_1 < \cdots < n_k$ have been found such that

$$\mathcal{R}(\bar{b}_{m_i}, \bar{c}_{n_j}) < r, \qquad \text{for } 1 \leq i < j \leq k;$$
$$\mathcal{R}(\bar{b}_{m_i}, \bar{c}_{n_j}) > s, \qquad \text{for } 1 \leq j < i \leq k.$$

Then we can find m_{k+1} and n_{k+1} such that

$$\mathcal{R}(\bar{b}_{m_i}, \bar{c}_{n_{k+1}}) < r, \qquad \text{for } 1 \leq i \leq k \text{ (by (ii))};$$
$$\mathcal{R}(\bar{b}_{m_{k+1}}, \bar{c}_{n_j}) > s, \qquad \text{for } 1 \leq j \leq k. \text{ (by (i))}.$$

(2) $\Rightarrow$ (1): Suppose that T is not stable. Then there exist a model E of T, a number $N > 0$, and a pair of formulas $\theta < \theta'$ which orders a sequence $(\bar{a}_n)$ in $\mathcal{B}_N$ i.e.,

$$E \models_{\mathcal{A}} \theta(\bar{a}_m, \bar{a}_n), \qquad \text{if } m \leq n;$$
$$E \models_{\mathcal{A}} \operatorname{neg}(\theta'(\bar{a}_m, \bar{a}_n)), \qquad \text{if } m > n.$$

By the Löwenheim-Skolem Theorem, we can assume that E is separable.

Take a set $\{\, \theta_r \mid, r \in \mathbb{Q}^+ \,\}$ of approximations of θ such that $\theta_r < \theta_s$ if and only if $r < s$, and $\theta_r \to \theta$ as $r \to 0$. Define

$$\mathcal{R}(\bar{a}, \bar{b}) = \inf\{\, r \in \mathbb{Q}^+ \mid \mathfrak{E} \models \theta_r(\bar{a}, \bar{b}) \in p \,\}.$$

The real-valued relation $\mathcal{R}$ is obviously definable. However, there exists $r > 0$ such that

$$\mathcal{R}(\bar{a}_m, \bar{a}_n) \leq 0 \qquad \text{if } m \leq n;$$
$$\mathcal{R}(\bar{a}_m, \bar{a}_n) \geq r \qquad \text{if } m > n.$$

Thus, if the limits of condition (2) exist, they cannot be equal. ⊣

The real-valued functions on Banach spaces which satisfy condition (3) of Theorem 9.1 were studied and characterized by Y. Raynaud in [12] and [11], where these functions are called “stable”. This terminology was motivated by the class Banach spaces introduced by J.-L. Krivine and B. Maurey in [9]. In this famous paper, the authors called *stable* the Banach spaces that satisfy condition (3) of Theorem 9.1 for the real-valued function $\mathcal{R}(x, y) = \|x + y\|$. This is, of course, a definable real-valued relation, so every stable Banach space is Krivine-Maurey stable. The main result of [9] is that every Krivine-Maurey stable Banach space contains the space ℓ_p, for some $1 \leq p < \infty$, almost isometrically. The problem of what Banach spaces contain copies of the classical sequence spaces has been central in functional analysis for several decades. The book [3] contains a fairly recent account of the body of work that has evolved around this question.

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