

STABLE MODELS AND REFLEXIVE BANACH SPACES

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ABSTRACT. We show that a formula $\varphi(x, y)$ is stable if and only if φ is the pairing map on the unit ball of $E \times E^*$, where E is a reflexive Banach space. The result remains true if the formula φ is replaced by a set of formulas $p(\bar{x}, \bar{y})$.

1. INTRODUCTION

In areas of mathematics where the object of study is a class of mathematical structures, one wishes to classify the structures in the class by drawing dividing lines between the simpler and the more complex structures of the class. The purpose of this paper is to point out a rather striking analogy between classification programs in two fields of mathematics which one does not normally regard as being closely related: model theory and Banach space theory.

In model theory, a clear dividing line is recognized between two kinds of models: stable and unstable models. A natural measure of the complexity of a model is given by its space of types, and a stable model is one whose space of types is not much larger than the model itself.

A similar distinction exists in Banach space geometry between reflexive and nonreflexive spaces. A Banach space is reflexive if it equals its double dual. An equivalent formulation is that Banach space is reflexive if and only if its unit ball is weakly compact. Intuitively, this can be taken to mean that the unit ball of dual of the space is not much larger than the unit ball of the space itself.

In 1964, R. C. James proved the following criterion for reflexivity [7].

Theorem (James Condition for Reflexivity). *The following conditions are equivalent for a Banach space E .*

- (1) E is not reflexive;
- (2) For every $\theta \in \mathbb{R}$ with $0 < \theta < 1$ there exists a sequence (a_m) with $\|a_m\| = 1$ for every m and a sequence (f_n) of linear functionals with $\|f_n\| = 1$ for every n , such that

$$f_n(a_m) = \begin{cases} 0 & \text{if } m < n \\ \theta & \text{if } m \geq n. \end{cases}$$

Such a characterization must capture the attention of a model theorist, for it bears striking resemblance to the most familiar characterization of model theoretical stability, due to S. Shelah [12]: a model M is unstable if and only if there exists a formula $\varphi(\bar{x}, \bar{y})$ and sequences (\bar{a}_m) and (\bar{b}_m) such that

$$M \models \varphi(\bar{a}_m, \bar{b}_n) \quad \text{if and only if} \quad m \leq n.$$

The formula φ is said to have the *order property*.

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During the 1970's, further classifying properties were identified, independently and in a parallel manner, in Banach space theory and in model theory. (For short essays on each of these programs, see [6] and [2]). Here we concentrate only on stability and reflexivity.

The similarity between (1) and (2) did not go unnoticed. In the early 1980's, J. L. Krivine and B. Maurey introduced the notion of *stable Banach space* in [9], and proved that every stable Banach space contains some ℓ_p almost isometrically, generalizing a result of D. Aldous [1] about subspaces of L_1 . A key fact in the Krivine-Maurey proof was noticing the analogy between the role played by random measures in Aldous' proof and that played by types in model theoretical stability.

Stable Banach spaces have now become objects of intense study.

Every occurrence of (2) is a particular case of (1): if E is nonreflexive, the formula

$$\varphi(x, y): "x \in B_E \text{ and } y \in B_{E^*} \text{ and } y(x) = 0"$$

(where B_E and B_{E^*} denote the unit balls of E and E^* respectively) has the order property in an appropriate structure.

In this paper we show the converse. Let us identify formulas with $\{0, 1\}$ -valued functions.

Theorem. *Suppose that M is a model and $\varphi(x, y)$ is a formula which does not have the order property on M . Then there exists a reflexive Banach space E and a map $(u, v): M \times M \rightarrow B_E \times B_{E^*}$ such that the diagram*

$$\begin{array}{ccc} M \times M & \xrightarrow{(u, v)} & B_E \times B_{E^*} \\ & \searrow \varphi & \swarrow \text{Evaluation map} \\ & [0, 1] & \end{array}$$

commutes.

The theorem is also true, and the proof is the same, if the formula $\varphi(x, y)$ above is replaced by a set of formulas $p(\bar{x}, \bar{y})$.

This result is actually a simple consequence of a lemma of Y. Raynaud [11] (Proposition 1.1) which comes from his thesis [10]. The lemma in question is a generalization of Theorem II.1 of [9], where only separable spaces are considered, to nonseparable spaces.

The connection of the Krivine-Maurey-Raynaud result with model theoretical stability is in fact simple, but we do not believe that it is known to the model theory community. Here we present the complete proof. The presentation should be accessible to a reader who has had a basic course in functional analysis. No knowledge of model theory will be assumed, but the reader familiar with model theory will recognize several connections.

2. STABLE FORMULAS AND STABLE FUNCTIONS

We identify formulas with $\{0, 1\}$ -valued functions defined on models. Thus, a formula $\varphi: M \times M \rightarrow \{0, 1\}$ has the order property if there exist sequences (a_m) and (b_n) in M such that

$$\varphi(a_m, b_n) = 1 \quad \text{if and only if} \quad m \leq n.$$

Definition 2.1. We will say that a formula $\varphi(x, y)$ *unstable* if either φ or $\neg\varphi$ has the order property. We will call φ *stable* if φ is not unstable.

Now we recall the notion of convergence relative to ultrafilters. Let X be a topological space and let $(a_i)_{i \in I}$ be an indexed family in X . If a is a limit point in X and \mathcal{U} is an ultrafilter on I , we write

$$(1) \quad \lim_{i, \mathcal{U}} a_i = a$$

if for every neighborhood O of a there exists $U \in \mathcal{U}$ such that $a_i \in O$ for every $i \in U$. If A is a subset of X , then $a \in \bar{A}$ if and only if there exists $(a_i)_{i \in I}$ in A and an ultrafilter \mathcal{U} on I such that (1) holds. The space X is compact if and only if $\lim_{i, \mathcal{U}} a_i$ exists for every family $(a_i)_{i \in I}$ in X . In particular, every bounded family of real numbers has a limit with respect to any ultrafilter.

Proposition 2.2. *Let M be a model and let $\varphi(x, y)$ be a formula on M . The following conditions are equivalent.*

- (1) φ is stable on M ;
- (2) If (a_m) and (b_n) are sequences in M and \mathcal{U}, \mathcal{V} are ultrafilters on \mathbb{N} , we have

$$\lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \varphi(a_m, b_n) = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} \varphi(a_m, b_n).$$

Proof. (1) \Rightarrow (2) is trivial. (2) \Rightarrow (1) follows from Proposition 2.4 below. \dashv

The preceding result motivates the following definition.

Definition 2.3. We will say that a bounded function $\varphi: A \times B \rightarrow \mathbb{R}$ is *stable* if the following condition holds. Whenever (a_m) is a sequence in A and (b_n) is a sequence in B , and \mathcal{U}, \mathcal{V} are ultrafilters on \mathbb{N} ,

$$\lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \varphi(a_m, b_n) = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} \varphi(a_m, b_n).$$

The following proposition shows that the sequences (a_m) and (b_n) in the preceding definition can be replaced by arbitrary families.

Proposition 2.4. *Let $\varphi: A \times B \rightarrow [0, 1]$. Let $(a_i)_{i \in I}$ be a family in A and $(b_j)_{j \in J}$ be a family in B . Suppose that*

$$\lim_{i, \mathcal{U}} \lim_{j, \mathcal{V}} \varphi(a_i, b_j) = \alpha, \quad \lim_{j, \mathcal{V}} \lim_{i, \mathcal{U}} \varphi(a_i, b_j) = \beta.$$

then there exist sequences (a_{i_m}) and (b_{j_n}) such that

$$\lim_{\substack{m < n \\ m \rightarrow \infty}} \varphi(a_{i_m}, b_{j_n}) = \alpha, \quad \lim_{\substack{n < m \\ n \rightarrow \infty}} \varphi(a_{i_m}, b_{j_n}) = \beta.$$

Proof. By definition, for every $\epsilon > 0$ there exist $U_\epsilon \in \mathcal{U}$ and $V_\epsilon \in \mathcal{V}$ such that

$$\begin{aligned} i \in U_\epsilon & \text{ implies } |\lim_{j, \mathcal{V}} \varphi(a_i, b_j) - \alpha| \leq \epsilon, \\ j \in V_\epsilon & \text{ implies } |\lim_{i, \mathcal{U}} \varphi(a_i, b_j) - \beta| \leq \epsilon. \end{aligned}$$

Also, for every $\hat{i} \in I$, every $\hat{j} \in J$ and every $\epsilon > 0$ there exist $V_\epsilon^{\hat{i}} \in \mathcal{V}$ and $U_\epsilon^{\hat{j}} \in \mathcal{U}$ such that

$$\begin{aligned} j \in V_\epsilon^{\hat{i}} & \text{ implies } |\varphi(a_{\hat{i}}, b_j) - \lim_{j, \mathcal{V}} \varphi(a_{\hat{i}}, b_j)| \leq \epsilon, \\ i \in U_\epsilon^{\hat{j}} & \text{ implies } |\varphi(a_i, b_{\hat{j}}) - \lim_{i, \mathcal{U}} \varphi(a_i, b_{\hat{j}})| \leq \epsilon. \end{aligned}$$

To construct the desired subsequences, take $i_0, i_1 < \dots$ and $j_0 < j_1 < \dots$ such that

$$\begin{aligned} i_0 &\in U_1 \\ j_0 &\in V_1 \cap V_1^{i_0} \\ i_1 &\in U_{1/2} \cap V_{1/2}^{j_0} \\ j_1 &\in V_{1/2} \cap V_{1/2}^{i_0} \cap V_{1/2}^{i_1} \\ &\vdots \\ i_N &\in U_{1/2^N} \cap \bigcap_{k < N} U_{1/2^k}^{j_k} \\ j_N &\in V_{1/2^N} \cap \bigcap_{k \leq N} V_{1/2^k}^{i_k}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} m < n &\quad \text{implies} \quad |\varphi(a_{i_m}, b_{j_n}) - \alpha| \leq 1/2^{m-1}, \\ n < m &\quad \text{implies} \quad |\varphi(a_{i_m}, b_{j_n}) - \beta| \leq 1/2^{n-1}. \end{aligned}$$

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Definition 2.5. Let $\varphi: A \times B \rightarrow [0, 1]$. If $a \in A$, the *left φ -type* of a , denoted $\text{lt}_\varphi(a)$ is the function $y \mapsto \varphi(a, y)$. Similarly, if $b \in B$, the *right φ -type* of b , denoted $\text{rt}_\varphi(b)$ is the function $x \mapsto \varphi(x, b)$.

The *space of left φ -types*, denoted $\text{LS}(\varphi)$, is the closure of

$$\{\text{lt}_\varphi(a) \mid a \in A\}$$

in $[0, 1]^B$ with respect to the product topology. The space $\text{RS}(\varphi)$ of *right φ -types* is the closure of

$$\{\text{rt}_\varphi(b) \mid b \in B\}$$

in $[0, 1]^A$. The spaces $\text{LS}(\varphi)$ and $\text{RS}(\varphi)$ are, of course, compact.

Proposition 2.6. *The following conditions are equivalent for any bounded function $\varphi: A \times B \rightarrow [0, 1]$.*

- (1) φ is stable;
- (2) There is a separately continuous function $\hat{\varphi}: \text{LS}(\varphi) \times \text{RS}(\varphi) \rightarrow [0, 1]$ such that

$$\hat{\varphi}(\text{lt}_\varphi(a), \text{rt}_\varphi(b)) = \varphi(a, b)$$

for $(a, b) \in A \times B$.

Condition (2) can be represented schematically by the following commutative diagram.

$$\begin{array}{ccc} \text{LS}(\varphi) \times \text{RS}(\varphi) & \xrightarrow{\hat{\varphi}} & [0, 1] \\ \uparrow (\text{lt}_\varphi, \text{rt}_\varphi) & \nearrow \varphi & \\ A \times B & & \end{array} \quad (\varphi \text{ stable} \implies \hat{\varphi} \text{ separately continuous})$$

Proof. (1) \Rightarrow (2): Take types $p \in \text{LS}(\varphi)$ and $q \in \text{RS}(\varphi)$.

Then there exists families $(a_i)_{i \in I}$ in A and $(b_j)_{j \in J}$ in B , and ultrafilters \mathcal{U} on I and \mathcal{V} on J such that

$$p = \lim_{i, \mathcal{U}} \text{ltp}_\varphi(a_i), \quad q = \lim_{j, \mathcal{V}} \text{rtp}_\varphi(b_j).$$

By Proposition 2.4,

$$\lim_{i, \mathcal{U}} \lim_{j, \mathcal{V}} \varphi(a_i, b_j) = \lim_{j, \mathcal{V}} \lim_{i, \mathcal{U}} \varphi(a_i, b_j).$$

Define $\hat{\varphi}(p, q)$ as the above common value. If we show that this definition is independent of the choice of (a_i) , (b_j) and \mathcal{U} , \mathcal{V} , the above equality will prove that $\hat{\varphi}$ is separately continuous.

Suppose that

$$p = \lim_{i, \mathcal{U}} \text{ltp}_\varphi(a_i) = \lim_{i, \mathcal{U}'} \text{ltp}_\varphi(a'_i), \quad q = \lim_{j, \mathcal{V}} \text{rtp}_\varphi(b_j) = \lim_{j, \mathcal{V}'} \text{rtp}_\varphi(b'_j).$$

Then,

$$\begin{aligned} \lim_{j, \mathcal{V}} \varphi(x, b_j) &= q(x) = \lim_{j, \mathcal{V}'} \varphi(x, b'_j), & \text{for every } x \in A, \\ \lim_{i, \mathcal{U}} \varphi(a_i, y) &= p(y) = \lim_{i, \mathcal{U}'} \varphi(a'_i, y), & \text{for every } y \in B. \end{aligned}$$

Therefore,

$$\lim_{i, \mathcal{U}} \lim_{j, \mathcal{V}} \varphi(a_i, b_j) = \lim_{i, \mathcal{U}} \lim_{j, \mathcal{V}'} \varphi(a_i, b'_j) = \lim_{j, \mathcal{V}'} \lim_{i, \mathcal{U}} \varphi(a_i, b'_j) = \lim_{j, \mathcal{V}'} \lim_{i, \mathcal{U}'} \varphi(a'_i, b'_j).$$

(2) \Rightarrow (1): Take sequences (a_n) in A and (b_j) in B , and ultrafilters \mathcal{U} and \mathcal{V} on \mathbb{N} . Let

$$p = \lim_{i, \mathcal{U}} \text{ltp}_\varphi(a_m), \quad q = \lim_{n, \mathcal{V}} \text{rtp}_\varphi(b_n).$$

Then $(p, q) \in \text{LS}(\varphi) \times \text{RS}(\varphi)$ and

$$\lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \varphi(a_m, b_n) = \hat{\varphi}(p, q) = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} \varphi(a_m, b_n).$$

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In Section 3, we will use the following lemma from real analysis. The proof is taken from [9].

Lemma 2.7. *Every separately continuous real-valued function on a product of compact metrizable spaces is a pointwise limit of a sequence of continuous functions.*

Proof. Let $f: K \times L \rightarrow \mathbb{R}$ be a separately continuous function, where K and L are compact metrizable spaces. We construct a sequence (f_n) of continuous functions which converges to f pointwise. Let d be a metric compatible with the topology of K . For $n \in \mathbb{N}$ and $a \in K$, define a function u_a^n on K by

$$u_a^n(x) = \begin{cases} \frac{1}{n} - d(x, a) & \text{if } d(x, a) \leq \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{N}$, fix a finite sequence $(a_i^n)_{i \in I(n)}$ in K such that every $x \in K$ is within $1/n$ of some a_i^n . Define a function v_i^n on K by

$$v_i^n(x) = \frac{u_{a_i^n}^n(x)}{\sum_{j \in I(n)} u_{a_j^n}^n(x)}.$$

Then, v_i^n is continuous, $|v_i^n| \leq 1$, and $\sum_{i \in I(n)} v_i^n = 1$. Now define f_n on $K \times L$ by

$$f_n(x, y) = \sum_{i \in I(n)} v_i^n(x) f(a_i^n, y).$$

We now show that (f_n) converges to f pointwise. Fix $(x, y) \in K \times L$. For each $n \in \mathbb{N}$, find $i(x, n) \in I_n$ such that $d(x, a_{i(x, n)}) \leq 1/n$. Take $N \in \mathbb{N}$ such that

$$|f(x, y) - f(a_{i(x, n)}, y)| \leq \epsilon, \quad \text{for } n \geq 1.$$

Then, for $n \geq 1$, we have

$$\begin{aligned} |f(x, y) - f_n(x, y)| &= \left| \sum_{i \in I(n)} f(x, y) - f(a_i^n, y) \right| v_i^n(x) \\ &\leq \sum_{i \in I(n)} |f(x, y) - f(a_i^n, y)| v_i^n(x) \\ &\leq |f(x, y) - f(a_{i(x, n)}, y)| v_{i(x, n)}^n(x) \leq \epsilon. \end{aligned}$$

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3. LINEARIZATION OF STABLE FUNCTIONS

Let E be a Banach space and let E^* be its dual. If $x \in E$ and $y \in E^*$, it is customary to write $\langle x, y \rangle$ for $y(x)$. The *weak topology* on E is the smallest topology on E for which the all the maps $x \mapsto \langle x, y \rangle$ are continuous. In other words, if $(x_i)_{i \in I}$ is a family in E and \mathcal{U} is an ultrafilter on I , we have $x = \lim_{i \in \mathcal{U}} x_i$ in the weak topology if and only if $\langle x, y \rangle = \lim_{i \in \mathcal{U}} \langle x_i, y \rangle$ for every $y \in E^*$. The *weak** topology on E^* is the smallest topology on E^* for which all the maps $y \mapsto \langle x, y \rangle$ are continuous. Thus, $(y_i)_{i \in I}$ is a family in E^* and \mathcal{U} is an ultrafilter on I , we have $y = \lim_{i \in \mathcal{U}} y_i$ in the weak* topology if and only if $\langle x, y \rangle = \lim_{i \in \mathcal{U}} \langle x, y_i \rangle$ for every $x \in E$. Alaoglu's theorem states that the unit ball of E^* is weak*-compact. The weak and weak* topologies are denoted $\sigma(E, E^*)$ and $\sigma(E^*, E)$, respectively.

Every $x \in E$ defines naturally a linear functional \tilde{x} on E^* , namely, $\langle y, \tilde{x} \rangle = \langle x, y \rangle$, for $y \in E^*$. The map $x \mapsto \tilde{x}$ is an isometric embedding of E into E^{**} , and is called the *canonical embedding*. We say that E is *reflexive* if the canonical embedding is surjective. In general, the canonical embedding is not surjective and E^{**} is larger than E . However, the following is always true.

Fact 3.1. *The unit ball of E is $\sigma(E^{**}, E^*)$ -dense in the unit ball of E^{**} .*

The simplest example of a nonreflexive space is the space c_0 of all the zero-convergent sequences with the supremum norm. The double dual of c_0 is ℓ^∞ , the space of all bounded sequences with the supremum norm. We now describe the dual of ℓ^∞ .

In general, for a set X let $\ell^\infty(X)$ the set of all bounded real-valued functions on X . $\ell^\infty(X)$ with the supremum norm is a Banach space. The dual of ℓ^∞ is the

space $\mathcal{M}(X)$ of all finitely additive set functions (or *measures*) of bounded variation, which we describe briefly below.

A finitely additive set function on X is a function $\mu: \mathcal{P}(X) \rightarrow \mathbb{R}$ such that

$$\mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i) = \mu(X), \quad \text{for } A_i \text{ pairwise disjoint.}$$

If μ is such a function, the *variation* $V(\mu)$ of μ is defined

$$V(\mu) = \sup \left\{ \sum_{i=1}^n |\mu(A_i)| \mid \bigcup_{i=1}^n A_i, A_i \text{ pairwise disjoint} \right\}.$$

We say that μ is of *bounded variation* if $V(\mu)$ is finite.

We denote by $\mathcal{M}(X)$ the space of all finitely additive functions of bounded variation on X with addition and scalar multiplication defined in the natural way, and the norm given by V . The space $\mathcal{M}(X)$ is the dual of $\ell^\infty(X)$. For a proof, see [8]. If $f \in \ell^\infty(X)$ and $\mu \in \mathcal{M}(X)$, $\langle f, \mu \rangle$ is also written

$$\int_X f d\mu.$$

For each $x \in X$, let δ_x be the element of $\mathcal{M}(X)$ defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

δ_x is called the *Dirac measure concentrated at x* . If $f \in \ell^\infty(X)$,

$$\int_X f d\delta_x = f(x).$$

If $(\xi_i)_{i \in I}$ is a family of nonnegative real numbers, the *sum* $\sum_{i \in I} \xi_i$ of $(\xi_i)_{i \in I}$ is defined as the supremum of all finite sums $\sum_{k=1}^n \xi_{i_k}$, where $i_1, \dots, i_k \in I$. Notice that if $\sum_{i \in I} \xi_i$ is finite, then $(\xi_i)_{i \in I}$ must have countable support.

If X is any set, the Banach space $\ell^1(X)$ is defined as follows. The elements of X are the functions $f: X \rightarrow \mathbb{R}$ such that $\sum_{x \in X} |f(x)|$ is finite. If $f \in \ell^1(X)$, the norm of f is given by the above sum. The dual of $\ell^1(X)$ is $\ell^\infty(X)$. The canonical embedding of $\ell^1(X)$ in $\ell^\infty(X)^* = \mathcal{M}(X)$ is given by

$$f \mapsto \sum_{x \in X} f(x) \delta_x.$$

Notation. The unit ball of a Banach space E will be denoted B_E .

Proposition 3.2. *Let $\varphi: A \times B \rightarrow [0, 1]$ be a stable function. Then,*

- (1) *For every $\mu \in \ell^\infty(A)^*$ and every $\nu \in \ell^\infty(B)^*$, we have*

$$\iint \varphi(x, y) d\mu(x) d\nu(y) = \iint \varphi(x, y) d\nu(y) d\mu(x).$$

- (2) *If $F(\mu, \nu)$ denotes the above common value, then the function F is weakly*-continuous on $\ell^\infty(A)^* \times \ell^\infty(B)^*$.*

Proof. First, we notice that the equality in (1) holds for $\mu \in \ell^1(A)$ and every $\nu \in \ell^1(B)$. Indeed, if $\mu = \sum_{i=1}^{\infty} \xi_i \delta_{a_i}$ and $\nu = \sum_{j=1}^{\infty} \eta_j \delta_{b_j}$, we have

$$\iint \varphi(x, y) d\mu(x) d\nu(y) = \sum_{i,j \in \mathbb{N}} \xi_i \eta_j \varphi(a_i, b_j) = \iint \varphi(x, y) d\nu(y) d\mu(x).$$

Define $F: \ell^1(A) \times \ell^1(B) \rightarrow \mathbb{R}$ by letting $F(\mu, \nu)$ be the above common value. We now wish to extend F to $\ell^\infty(A)^* \times \ell^\infty(B)^*$.

Claim. The restriction of F to $B_{\ell^1(A)} \times B_{\ell^1(B)}$ is stable.

Proof of the claim. Let (μ_m) be a sequence in $B_{\ell^1(A)}$ and let (ν_n) be a sequence in $B_{\ell^1(B)}$. Let

$$S = \bigcup_m \text{supp}(\mu_m), \quad T = \bigcup_n \text{supp}(\nu_n).$$

Then S and T are countable. We now apply Proposition 2.6 to the restriction $\varphi|_{S \times T}$. Let $\text{LS}(\varphi|_{S \times T})$ and $\text{RS}(\varphi|_{S \times T})$ be spaces of left and right φ -types, respectively (see Definition 2.5). These spaces are compact and metrizable, so by Proposition 2.6, there exists a separately continuous function $\hat{\varphi}: \text{LS}_{\varphi|_{S \times T}} \times \text{RS}_{\varphi|_{S \times T}}$ which “extends” φ in the sense that for $(a, b) \in S \times T$,

$$\hat{\varphi}(\text{lt}_{\varphi|_{S \times T}}(a), \text{rt}_{\varphi|_{S \times T}}(b)) = \varphi(a, b).$$

By Lemma 2.7, $\hat{\varphi}$ is a Borel function. This allows us to define a function \hat{F} as follows. If μ is a measure on $\text{LS}_{\varphi|_{S \times T}}$ and ν is a measure on $\text{RS}_{\varphi|_{S \times T}}$,

$$\hat{F}(\mu, \nu) = \iint_{\text{LS}_{\varphi|_{S \times T}} \times \text{RS}_{\varphi|_{S \times T}}} \hat{\varphi}(p, q) d\mu(p) d\nu(q).$$

Now, \hat{F} “extends” F in the sense that if

$$\mu = \sum_{i=1}^{\infty} \xi_i \delta_{a_i} \in \ell^1(A) \quad \nu = \sum_{j=1}^{\infty} \eta_j \delta_{b_j} \in \ell^1(B),$$

and we let

$$\hat{\mu} = \sum_{i=1}^{\infty} \xi_i \delta_{\text{lt}_{\varphi|_{S \times T}}(a_i)}, \quad \hat{\nu} = \sum_{j=1}^{\infty} \eta_j \delta_{\text{rt}_{\varphi|_{S \times T}}(b_j)},$$

we get

$$\hat{F}(\hat{\mu}, \hat{\nu}) = \sum_{i,j \in \mathbb{N}} \xi_i \eta_j \varphi(a_i, b_j) = F(\mu, \nu).$$

By Fubini’s theorem,

$$\begin{aligned} (*) \quad \hat{F}(\mu, \nu) &= \int_{\text{LS}_{\varphi|_{S \times T}}} \int_{\text{RS}_{\varphi|_{S \times T}}} \hat{\varphi}(p, q) d\nu(q) d\mu(p) \\ &= \int_{\text{RS}_{\varphi|_{S \times T}}} \int_{\text{LS}_{\varphi|_{S \times T}}} \hat{\varphi}(p, q) d\mu(p) d\nu(q). \end{aligned}$$

and \hat{F} is separately weakly*-continuous.

Take now measures μ and ν such that

$$\begin{aligned}\mu &= \lim_m \hat{\mu}_m, & \text{in } \sigma(\ell^\infty(LT_{\varphi|S \times T})^*, \ell^\infty(LT_{\varphi|S \times T})); \\ \nu &= \lim_n \hat{\nu}_n, & \text{in } \sigma(\ell^\infty(RT_{\varphi|S \times T})^*, \ell^\infty(RT_{\varphi|S \times T})).\end{aligned}$$

Then, by (*),

$$\lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} F(\mu_m, \nu_n) = \lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \hat{F}(\hat{\mu}_m, \hat{\nu}_n) = \hat{F}(\mu, \nu) = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} \hat{F}(\hat{\mu}_m, \hat{\nu}_n) = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} F(\mu_m, \nu_n).$$

Since (μ_m) and (ν_n) are arbitrary, the claim is proved. \dashv

Now we use Fact 3.1 and the previous claim to extend F to $\ell^\infty(A)^* \times \ell^\infty(B)^*$. Take $\mu \in B_{\ell^\infty(A)^*}$ and $\nu \in B_{\ell^\infty(B)^*}$. Take families $(\mu_i)_{i \in I}$ in $B_{\ell^1(A)}$ and $(\nu_j)_{j \in J}$ in $B_{\ell^1(B)}$, and ultrafilters \mathcal{U} on I and \mathcal{V} on J such that

$$\begin{aligned}\mu &= \lim_{i, \mathcal{U}} \mu_i, & \text{in } \sigma(\ell^\infty(A)^*, \ell^\infty(A)); \\ \nu &= \lim_{j, \mathcal{V}} \nu_j, & \text{in } \sigma(\ell^\infty(B)^*, \ell^\infty(B)).\end{aligned}$$

Then,

$$\begin{aligned}\iint \varphi(x, y) d\nu(x) d\mu(y) &= \lim_{i, \mathcal{U}} \lim_{j, \mathcal{V}} \iint \varphi(x, y) d\mu_i(x) d\nu_j(y) \\ &= \lim_{i, \mathcal{V}} \lim_{j, \mathcal{U}} \iint \varphi(x, y) d\nu_j(x) d\mu_i(y) \\ &= \iint \varphi(x, y) d\nu(y) d\mu(x).\end{aligned}$$

This proves the theorem. \dashv

Now we prove the main result. Recall that an operator $T: X \rightarrow Y$ is *weakly compact* if the image of the unit ball in X is relatively $\sigma(Y, Y^*)$ -compact in Y . See [4].

Theorem 3.3. *Let $\varphi: A \times B \rightarrow [0, 1]$. Then the following conditions are equivalent.*

- (1) φ is stable;
- (2) There exists a reflexive Banach space E and a map $(u, v): A \times B \rightarrow B_E \times B_{E^*}$ such that $\varphi(x, y) = \langle u(x), v(y) \rangle$.

Proof. (1) \Rightarrow (2): Define an operator $T: \ell^\infty(A)^* \rightarrow \ell^\infty(B)$ by

$$T\mu(y) = \int \varphi(x, y) \mu(x)$$

and an operator $T^*: \ell^\infty(B)^* \rightarrow \ell^\infty(A)$ by

$$T^*\nu(x) = \int \varphi(x, y) \nu(y).$$

By the preceding proposition, if $\mu \in \ell^\infty(A)^*$ and $\nu \in \ell^\infty(B)^*$,

$$\langle T\mu, \nu \rangle = \iint \varphi(x, y) d\mu(x) d\nu(y) = \iint \varphi(x, y) d\nu(x) d\mu(y) = \langle T^*\nu, \mu \rangle.$$

We show that T is weakly compact, i.e., the image of the unit ball of $\ell^\infty(A)^*$ under T is relatively $\sigma(\ell^\infty(B), \ell^\infty(B)^*)$ -compact in $\ell^\infty(B)$. By the Eberlein-Smulian theorem (see, for example, [3]), it suffices to prove that every sequence in $T[B_{\ell^\infty(A)^*}]$ has a $\sigma(\ell^\infty(B), \ell^\infty(B)^*)$ -convergent subsequence.

Let (g_n) be a sequence in $T[B_{\ell^\infty(A)^*}]$. Take (μ_n) in $B_{\ell^\infty(A)^*}$ such that $T\mu_n = g_n$. Then, if $\nu \in \ell^\infty(B)^*$,

$$\langle g_n, \nu \rangle = \langle T\mu_n, \nu \rangle = \langle T^*\nu, \mu_n \rangle.$$

By Alaoglu's theorem, (μ_n) has a $\sigma(\ell^\infty(A)^*, \ell^\infty(A))$ -convergent sequence (μ_{n_k}) . The above equation shows that then (g_{n_k}) is $\sigma(\ell^\infty(B), \ell^\infty(B)^*)$ -convergent.

Now we apply a result of W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński [5]: Every weakly compact operator factors through a reflexive Banach space. Hence, there exists a reflexive Banach space E and operators $U: \ell^\infty(A)^* \rightarrow E$ and $V: E \rightarrow \ell^\infty(B)$ such that $T = V \circ U$. If for $x \in A$ and $y \in B$ we let δ_x and δ_y be the corresponding Dirac measures, we get

$$\varphi(x, y) = \langle T\delta_x, \delta_y \rangle = \langle V \circ U\delta_x, \delta_y \rangle = \langle U\delta_x, V^*\delta_y \rangle.$$

The theorem now follows by defining, for $(x, y) \in A \times B$,

$$u(x) = U\delta_x, \quad v(y) = V^*\delta_y.$$

(2) \Rightarrow (1): First notice that if a space E is reflexive, then B_E is $\sigma(E, E^*)$ -compact, since $\sigma(E, E^*) = \sigma(E^{**}, E^*)$ and $B_{E^{**}}$ is $\sigma(E^{**}, E^*)$ -compact by Alaoglu's theorem. (The converse is also true, but we won't need it here.)

Assume that E and (u, v) are as in (2). Take sequences (a_m) in A , (b_n) in B , and ultrafilters \mathcal{U}, \mathcal{V} on \mathbb{N} . Let

$$\begin{aligned} a &= \lim_{m, \mathcal{U}} u(a_m), & \text{in } \sigma(E, E^*) \\ b &= \lim_{n, \mathcal{V}} v(b_n), & \text{in } \sigma(E^*, E). \end{aligned}$$

Then,

$$\lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \varphi(a_m, b_n) = \lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \langle u(a_m), v(b_n) \rangle = \langle a, b \rangle.$$

We obtain the same result if the order of the limits is reversed. Hence, φ is stable. \dashv

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