OMITTING UNCOUNTABLE TYPES, AND THE STRENGTH OF $[0,1]$-VALUED LOGICS

XAVIER CAICEDO AND JOSÉ N. IOVINO

ABSTRACT. We study a class of $[0,1]$-valued logics. The main result of the paper is a maximality theorem that characterizes these logics in terms of a model-theoretic property, namely, an extension of the omitting types theorem to uncountable languages.

INTRODUCTION

In [CK66], Chang and Keisler introduced a model-theoretic apparatus for logics with truth values in a compact Hausdorff space whose logical operations are continuous, calling it *continuous logic*. In this paper we focus on the special case when the truth-value space is the closed unit interval $[0,1]$. We call it *basic continuous logic*. The main result of the paper is a characterization of this logic in terms of a model-theoretic property, namely, an extension of the omitting types theorem to uncountable languages. This result generalizes a characterization of first-order logic due to Lindström [Lin78]. By restricting basic continuous logic to particular classes of structures we obtain analogous characterizations of $[0,1]$-valued logics that have been studied extensively, namely, Łukasiewicz-Pavelka logic [Pav79a, Pav79b, Pav79c] (see also Section 5.4 of [Háj98]) and the first-order continuous logic framework of Ben Yaacov and Usvyatsov [BYU10].

To make this more precise, consider the following logic $\mathcal{L}_0$. The semantics is given by the class of *continuous metric structures*, that is, metric spaces with uniformly continuous functions and uniformly continuous $[0,1]$-valued predicates, the distance being considered a distinguished predicate which replaces the identity relation. The sentences of $\mathcal{L}_0$ are $[0,1]$-valued and they are built as follows. The atomic formulas are the predicate symbols and the distance symbol applied to terms. The connectives are the Łukasiewicz implication ($x \to y = \min\{1 - x + y, 1\}$) and the Pavelka rational constants, i.e., for each rational r in the closed interval $[0,1]$ a constant connective with value r (these are sufficient to generate, as uniform limits, all continuous connectives). The quantifiers are $\forall$ and $\exists$ are interpreted as infima and suprema of truth-values, respectively (only one of them is needed).

We observe that the restriction of $\mathcal{L}_0$ to the class of discrete metric structures is first-order logic, its restriction to the class of 1-Lipschitz structures is predicate Łukasiewicz-Pavelka logic, and its restriction to the class of complete structures yields the continuous logic framework of [BYU10], called continuous logic in recent literature.

In general, a formula $\varphi(\bar{x})$ of an arbitrary $[0,1]$-valued logic $\mathcal{L}$ assigns to each structure M of its semantic domain and each tuple $\bar{a}$ in M a *truth-value* $\varphi^M(\bar{a})$ belonging to $[0,1]$. In this context, we may define a satisfaction relation: $M \models_{\mathcal{L}} \varphi[\bar{a}]$ if and only if $\varphi^M(\bar{a}) = 1$. Based on $\models_{\mathcal{L}}$, we introduce classical notions as consistency, semantical consequence, etc.

If λ is an uncountable cardinal and T is a theory of cardinality $\leq \lambda$ in a logic $\mathcal{L}$, we will say that a partial type $\Sigma(x)$ of $\mathcal{L}$ is λ-*principal* over T if there exists a set of formulas $\Phi(x)$ of cardinality $< \lambda$ such that $T \cup \Phi(x)$ is consistent and $T \cup \Phi(x) \models_{\mathcal{L}} \Sigma(x)$. The notion of ω-principal is sligthly more involved (see Definition 3.4).

A logic $\mathcal{L}$ satisfies the λ-omitting types property if whenever T is a consistent theory of $\mathcal{L}$ of cardinality $\leq \lambda$ and $\{\Sigma_j(x)\}_{j<\lambda}$ is a set of types that are not λ-principal over T there is a model of T that omits each $\Sigma_j(x)$.

In the first part of the paper we prove the following result:

Theorem 1. $\mathcal{L}_0$ *satisfies the λ-omitting types property, for every infinite cardinal λ.*

In the second part we show that this property for uncountable cardinals characterizes $\mathcal{L}_0$:

Theorem 2. *Let $\mathcal{L}$ be a $[0,1]$-valued logic that extends $\mathcal{L}_0$ and satisfies the following properties:*

- *The λ-omitting types property for every uncountable cardinal λ,*
- *Closure under the of Łukasiewicz-Pavelka connectives* (see below) *and the existential quantifier,*
- *Every continuous metric structure is logically equivalent in $\mathcal{L}$ to its metric completion.*

Then every sentence in $\mathcal{L}$ is is a uniform limit of sentences in $\mathcal{L}_0$.

By restricting Theorem 2 to the class of 1-Lipschitz structures we obtain a characterization of Łukasiewicz-Pavelka logic, and by restricting it to the class of complete structures we obtain an analogous characterization of continuous logic. See Corollary 4.7. However, the latter case uses a form of the λ-omitting types property asserting that the type-omitting structure is complete. This version requires a stronger notion of type principality, but it follows from the λ-omitting types property of $\mathcal{L}_0$.

Our proof of the λ-omitting types property is based on a general version of the Baire category theorem (Proposition 3.2). The proof covers at once the uncountable case and, with a minor modification, the case $\lambda = \omega$. (See Theorem 3.6.) The countable case is not new; omitting types theorems for $[0,1]$-valued logics over countable languages have been proved by Murinová-Novák [MN06] (for Łukasziewicz-Pavelka logic) and by Henson [BYU07, BYBHU08] (for complete metric structures).

Our approach is topological. The usefulness of topological methods to study model-theoretic properties of abstract logics is at the heart of several papers by the first author (see, for example, [Cai93, Cai95, Cai99]). The topological approach followed in those papers is particularly well-suited for settings such as those considered here, where the logics at hand do not have negation in the classical sense. In such settings the assumption that the space of structures is topologically regular serves as the "correct" replacement of the classical negation. (Indeed, the collaboration between both authors originated with the realization that the concept of

regular logic, which was isolated by the first author, is equivalent to the concept of logics with weak negation that was introduced by the second author in [Iov01].) Utilizing these ideas, the first author has proved topological versions of Lindström's first theorem for first-order logic [Cai1, Cai2].

The paper is self-contained; no previous familiarity with abstract model theory, with Łukasiewicz logic, or with continuous logic is presumed. The basic definitions are given in Section 1. Section 2 introduces the topological preliminaries. Section 3 is devoted to the proof of the λ-omitting types property, and Section 4 contains the proof the Main Theorem, Theorem 2.

This paper evolved from a manuscript prepared by Jonathan Brucks within his master's degree research, under the supervision of both authors, during the academic year 2010–2011.

1. Structures and Logics

1.1. **Continuous metric structures.** Although, for simplicity, we will focus on $[0, 1]$-valued continuous metric structures, our results regarding continuous logic may be easily extended to bounded $\mathbb{R}$-valued structures, or even unbounded structures, if we decompose them into bounded ones (see Section 1.7.), thus we prefer to give the more general definition of continuous metric structure.

Definition 1.1. A *continuous metric structure* (or simply a *structure*) M consists of the following items:

(1) A family $(M_i, d_i)_{i \in I}$ of metric spaces.
(2) A collection of functions of the form

$$F : M_{i_1} \times \cdots \times M_{i_n} \to M_{i_0},$$

called the *operations* of the structure, each of which is uniformly continuous on every bounded subset of its domain.
(3) A collection of real-valued functions of the form

$$R : M_{i_1} \times \cdots \times M_{i_n} \to \mathbb{R},$$

called the *predicates* of the structure, each of which is uniformly continuous on every bounded subset of its domain.

The *constants* of a structure are the 0-ary operations of the structure.

The metric spaces M_i are called the *sorts* of M, and we say that M is *based on* $(M_i)_{i \in I}$. If M is based on $(M_i)_{i \in I}$, we will say that a structure is *discrete* if for each $i \in I$ the distinguished metric on the sort M_i is the discrete metric, and all the predicates of M take values in $\{0, 1\}$. Note that if M is a discrete structure, the uniform continuity requirement for the operations and predicates of M is superfluous. A structure is *complete* if all of its sorts are complete metric spaces.

Let M be a structure based on $(M_i)_{i \in I}$. If $(F_j)_{j \in J}$ is a list of the operations of M, and $(R_k)_{k \in K}$ is a list of the predicates of M, we may write

$$M = (M_i, F_j, R_k)_{i \in I, j \in J, k \in K}.$$

If $a_1, \ldots, a_n \in M_i$, we denote by $\bar{a}$ the list of elements $a_1, \ldots, a_n$ and write simply $\bar{a} \in M_i$. When the context allows it, we also denote by $\bar{a}$ denote the tuple $(a_1, \ldots, a_n)$ by $\bar{a}$. If it becomes necessary to refer to the length of a list or tuple of elements $\bar{a}$, we denote it by $\ell(\bar{a})$.

Examples of non-discrete structures include normed spaces, Banach algebras, Banach lattices, and operator spaces. For more examples, see [HI02, Examples 2.2] and [BYBHU08, Examples 2.1].

If a continuous metric structure M is based on a single metric space (M, d), we say that M is a *one-sorted structure*. In this case we call (M, d) the *universe* of M. For simplicity, we shall restrict our attention to one-sorted metric structures.

Note that, informally, we use the same letter to denote a structure and its universe. We follow this convention throughout the paper.

Let M be a continuous metric structure, and let N be a metric space that extends M and contains M as a dense subset. By the uniform continuity condition in Definition 1.1, each operation or predicate of M has a unique extension to an operation or predicate of N.

The *completion* of a continuous metric structure $(M, F_i, R_j)_{i \in I, j \in J}$ is the structure $(\overline{M}, \overline{F}_i, \overline{R}_j)_{i \in I, j \in J}$, where $\overline{M}$ is the metric completion of M and $\overline{F}_i, \overline{R}_j$ are the unique extensions of F_i, R_j from the appropriate powers of M to the corresponding powers of $\overline{M}$.

We call a one-sorted continuous metric structure *bounded* if its universe is bounded.

1.2. **Signatures.** In order to treat a metric structure M model-theoretically, it is convenient to have a formal way of indexing the operations and predicates of M, and specifying moduli of uniform continuity for them; this is a *signature* for M.

Definition 1.2. Let M be a bounded continuous metric structure with metric d. A *signature* for M is a pair $\mathbf{S} = (S, \mathcal{U})$, where:

(1) S is a first-order vocabulary consisting of the following items: for each operation $F : M^n \to M$ of M (respectively, predicate $R : M^n \to \mathbb{R}$ of M), a pair of the form (f, n) (respectively, (P, n)), where f and P are syntactic symbols called n-ary *operation symbol* and n-ary *predicate symbol*, respectively. If $n = 0$, f is called a *constant symbol*. In this context, F is denoted f^M and called the *interpretation of f* and R is denoted P^M and similarly called the *interpretation of P*.

(2) $\mathcal{U}$ is a family of *uniform continuity moduli* for the symbols of S, that is: $\mathcal{U}$ assigns to each each n-ary function symbol f or predicate symbol P of S an associated function $\delta : \mathbb{Q} \cap (0, 1) \to \mathbb{Q} \cap (0, 1)$ such that for every $\epsilon \in \mathbb{Q} \cap (0, 1)$ and all $\bar{a} = a_1, \ldots, a_n \in M$ and $\bar{b} = b_1, \ldots, b_n \in M$,

$$\sup_{1 \leq i \leq n} d(a_i, b_i) < \delta(\epsilon) \quad \Rightarrow \quad d(f^M(\bar{a}), f^M(\bar{b})) \leq \epsilon$$

in the first case and

$$\sup_{1 \leq i \leq n} d(a_i, b_i) < \delta(\epsilon) \quad \Rightarrow \quad |P^M(\bar{a}) - P^M(\bar{b})| \leq \epsilon$$

in the second.

If $\mathbf{S} = (S, \mathcal{U})$ is a signature for M, we say that M is an *S-structure* or an *$\mathbf{S}$-structure*, depending on whether the uniform continuity moduli of $\mathcal{U}$ need to be made explicit for the context.

If $\mathbf{S} = (S, \mathcal{U})$ and $\mathcal{U}$ assigns to each symbol of S the identity function, and $\mathbf{S}$ will be called a 1-*Lipschitz*-structure.

Note that if M is a Lipschitz S-structure, then $d(f^M(\bar{a}), f^M(\bar{b})), |P^M(\bar{a}) - P^M(\bar{b})| \leq \sup_{1 \leq i \leq n} d(a_i, b_i)$ for every n-ary function symbol f and predicate symbol P of S.

If S, S' are vocabularies, we write $S \subseteq S'$ if every operation and predicate symbol in S is in S' with the same arity. In this case we say that S' is an *extension* of S.

Let S, S' be vocabularies with $S \subseteq S'$ and suppose that N is an S'-structure. The *reduct* of N to S, denoted $N \upharpoonright S$, is the S-structure that results by removing from N the operations and predicates that are indexed by S' but not S. We say that a continuous metric structure N is an *expansion* of a structure M if M is a reduct of N.

Let S be a vocabulary and suppose that M and N are S-structures. We say that M is a *substructure* of N (or that N is an *extension* of M) if the following conditions hold:

· The universe of N contains the universe of M, and the distinguished metric of N extends the distinguished metric of M.
· For every operation symbol f of S, the operation f^N extends f^M.
· For every predicate symbol P of S, the predicate P^N extends P^M.

Let S be a vocabulary and let M, N be S-structures. A *metric isomorphism* between M and N is a surjective isometry $T : M \to N$ which commutes with the interpretation of the operation and predicate symbols of S. We say that M and N are *metrically isomorphic* (or simply *isomorphic*), and write $M \simeq N$, if there exists a metric isomorphism between M and N.

Clearly, if $\mathbf{S}$ is a signature and M is an $\mathbf{S}$-structure, then every structure that is metrically isomorphic to M is also an $\mathbf{S}$-structure.

A *renaming* is a bijection between vocabularies that sends function symbols to function symbols, predicate symbols to predicate symbols, and preserves arities. If M is an S-structure and $\rho : S \to S'$ is a renaming, we denote by M^ρ the S'-structure that results from converting M into an S'-structure through ρ.

1.3. **Logics.** The formal definition of model-theoretic logic was introduced by P. Lindström in his famous paper [Lin69]. Lindström's original definition of logic was intended for classical structures, i.e., discrete structures. Here we will use it for the more general context of continuous metric structure given in Definition 1.1. In general, throughout the paper, the word "structure" will stand for "continuous metric structure".

Definition 1.3. A *logic* $\mathcal{L}$ is a triple $(\mathcal{K}, \mathrm{Sent}_\mathcal{L}, \models_\mathcal{L})$, where $\mathcal{K}$ is a class of metric structures that is closed under under isomorphisms, renamings and reducts, $\mathrm{Sent}_\mathcal{L}$ is a function that assigns to every vocabulary S a set $\mathrm{Sent}_\mathcal{L}(S)$ called the set of S-*sentences of* $\mathcal{L}$, and $\models_\mathcal{L}$ is a binary relation between structures and sentences, such that the following conditions hold:

(1) If $S \subseteq S'$, then $\mathrm{Sent}_\mathcal{L}(S) \subseteq \mathrm{Sent}_\mathcal{L}(S')$.
(2) If $M \models_\mathcal{L} \varphi$ (i.e., if M and φ are related under $\models_\mathcal{L}$), then there is a vocabulary S such that M is an S-structure in $\mathcal{K}$ and φ an S-sentence.
(3) *Isomorphism Property.* If M, N are isomorphic structures in $\mathcal{K}$, then $N \models_\mathcal{L} \varphi$.

(4) *Reduct Property.* If $S \subseteq S'$, φ is an S-sentence, and M an S'-structure in $\mathcal{K}$, then

$$M \models_{\mathcal{L}} \varphi \qquad \text{if and only if} \qquad (M \upharpoonright S) \models_{\mathcal{L}} \varphi.$$

(5) *Renaming Property.* If $\rho : S \to S'$ is a renaming, then for each S-sentence φ there is an S'-sentence φ^ρ such that $M \models_{\mathcal{L}} \varphi$ if and only if $M^\rho \models_{\mathcal{L}} \varphi^\rho$ for every S-structure M in $\mathcal{K}$. (Recall that M^ρ denotes the structure that results from converting M into an S'-structure through ρ.)

If $M \models_{\mathcal{L}} \varphi$, we say that M *satisfies* φ, or that M is a *model* of φ.

The study of abstract logics is known as abstract model theory. For a survey, the reader is referred to [BF85].

A logic $\mathcal{L}$ is said to be *closed under conjunctions* if given any two $\mathcal{L}$-sentences φ, ψ there exists an $\mathcal{L}$-sentence $\varphi \wedge \psi$ such that for every structure M

$$M \models_{\mathcal{L}} \varphi \wedge \psi \qquad \text{if and only if} \qquad M \models_{\mathcal{L}} \varphi \quad \text{and} \quad M \models_{\mathcal{L}} \psi.$$

Similarly, $\mathcal{L}$ is said to be *closed under disjunctions* if given two $\mathcal{L}$-sentences φ, ψ there exists an $\mathcal{L}$-sentence $\varphi \vee \psi$ such that for every structure M

$$M \models_{\mathcal{L}} \varphi \vee \psi \qquad \text{if and only if} \qquad M \models_{\mathcal{L}} \varphi \quad \text{or} \quad M \models_{\mathcal{L}} \psi.$$

A logic $\mathcal{L}$ is said to be *closed under negations* if given an $\mathcal{L}$-sentence φ there exists an $\mathcal{L}$-sentence $\neg\varphi$ such that for every structure M

$$M \models_{\mathcal{L}} \neg\varphi \qquad \text{if and only if} \qquad M \nvDash_{\mathcal{L}} \varphi.$$

Abstract logics without negation have been studied in [Iov01, GM04, GMV05].

Convention 1.4. We will assume that all logics are closed under finite conjunctions and disjunctions, but not necessarily under negations. We will also assume that every logic $\mathcal{L}$ mentioned is nontrivial in the following sense: for every structure M, there is a sentence φ such that $M \nvDash_{\mathcal{L}} \varphi$.

Definition 1.5. Let $\mathcal{L}$ be a logic and let S be a vocabulary.

(1) An S-*theory* (or simply a *theory* if the vocabulary is given by the context) of $\mathcal{L}$ is a set of S-sentences of $\mathcal{L}$.
(2) Let T be an S-theory of $\mathcal{L}$. If M is an S-structure such that $M \models_{\mathcal{L}} \varphi$ for each $\varphi \in T$, we say that M is a *model* of T and write $M \models_{\mathcal{L}} T$.
(3) A theory is *consistent* if it has a model.

If $\varphi \in \mathrm{Sent}_{\mathcal{L}}(S)$ for some vocabulary S, but there is no need to refer to the specific vocabulary, we may refer to φ an $\mathcal{L}$-*sentence*. Similarly, when T is an S-theory for some vocabulary S and there is no need to refer to S, we may refer to T as an $\mathcal{L}$-*theory*.

If S is a vocabulary, $\bar{x} = x_1, \ldots, x_n$ is a finite list of constant symbols not in S, and φ is an $(S \cup \{\bar{x}\})$-sentence, we emphasize this by writing φ as $\varphi(\bar{x})$. In this case we may say that $\varphi(\bar{x})$ is an S-*formula*. If M is an S-structure and $\bar{a} = a_1, \ldots, a_n$ is a list of elements of M, we write

$$(M, a_1, \ldots, a_n) \models_{\mathcal{L}} \varphi(x_1, \ldots, x_n)$$

or

$$M \models_{\mathcal{L}} \varphi[\bar{a}]$$

if the $S \cup \{\bar{x}\}$ expansion of M that results from interpreting x_i by a_i (for $i = 1, \ldots, n$) satisfies $\varphi(\bar{x})$.

Definition 1.6. Let M, N be S-structures. We say that M and N are *equivalent in* $\mathcal{L}$, and write $M \equiv_{\mathcal{L}} N$, if for every S-sentence φ we have $M \models_{\mathcal{L}} \varphi$ if and only if $N \models_{\mathcal{L}} \varphi$.

If M is a structure and A is a subset of the universe of M, we denote by $(M, a)_{a \in A}$ the expansion of M that results by adding a constant symbol for each element of A. The structure $(M, a)_{a \in A}$ is said to be an *expansion of M by constants.*

Definition 1.7. Let $\mathcal{L}$ be a logic and let M, N be S-structures with M a substructure of N. We say that M is an *elementary substructure* of N (with respect to $\mathcal{L}$), and write $M \preceq_{\mathcal{L}} N$, if $(M, a)_{a \in A} \equiv (N, a)_{a \in A}$.

Recall that a signature is a pair $\mathbf{S} = (S, \mathcal{U})$, where S is a first-order vocabulary and $\mathcal{U}$ is a family of uniform continuity moduli for the symbols of S (see Definition 1.2).

Definition 1.8. Let $\mathcal{L}$ be a logic.
   (1) If λ is an infinite cardinal, $\mathcal{L}$ is *λ-compact* if whenever $\mathbf{S}$ is a signature and T is an S-theory of cardinality $\leq \lambda$ such that every finite subset of T is satisfied by an $\mathbf{S}$-structure, the theory T is satisfied by an $\mathbf{S}$-structure.
   (2) $\mathcal{L}$ is *compact* if it is λ-compact for every infinite λ.

The following concept will be needed for the statement of the Main Theorem (Theorem 4.1).

Definition 1.9. We say that a logic $\mathcal{L}$ has the *finite occurrence property* if for every vocabulary S and every S-sentence φ there is a finite vocabulary $S_0 \subseteq S$ such that φ is an S_0-sentence.

1.4. $[0, 1]$**-valued logics.** Hereafter, for simplicity, we focus on $[0, 1]$-valued structures, i.e., continuous metric structures where the distinguished metric and all the predicates take values on the closed unit interval $[0, 1]$. More general structures are discussed in Section 1.7.

We now refine Lindström's definition of logic (Definition 1.3):

Definition 1.10. A $[0, 1]$-*valued logic* is a triple $(\mathcal{K}, \mathrm{Sent}_{\mathcal{L}}, \mathcal{V})$, where $\mathcal{K}$ is a class of metric structures that is closed under under isomorphisms, renamings and reducts, $\mathrm{Sent}_{\mathcal{L}}$ is a function that assigns to every vocabulary S a set $\mathrm{Sent}_{\mathcal{L}}(S)$ called the set of S-*sentences of* $\mathcal{L}$, and $\mathcal{V}$ is a functional relation such that the following conditions hold:
   (1) If $S \subseteq S'$, then $\mathrm{Sent}_{\mathcal{L}}(S) \subseteq \mathrm{Sent}_{\mathcal{L}}(S')$.
   (2) The relation $\mathcal{V}$ assigns to every pair (φ, M), where φ is an S-sentence of $\mathcal{L}$ and M is an S-structure in $\mathcal{K}$, a real number $\varphi^M \in [0, 1]$ called the *truth value* of φ in M.
   (3) *Isomorphism Property for $[0, 1]$-valued logics.* If M, N are metrically isomorphic structures in $\mathcal{K}$ and φ is an S-sentence of $\mathcal{L}$, then $\varphi^M = \varphi^N$.
   (4) *Reduct Property for $[0, 1]$-valued logics.* If $S \subseteq S'$, φ is an S-sentence of $\mathcal{L}$, and M an S'-structure in $\mathcal{K}$, then $\varphi^M = \varphi^{M \restriction S}$.
   (5) *Renaming Property for $[0, 1]$-valued logics.* If $\rho : S \to S'$ is a renaming, then for each S-sentence φ of $\mathcal{L}$ there is an S'-sentence φ^ρ such that $\varphi^M = (\varphi^\rho)^{M^\rho}$ for every S-structure M in $\mathcal{K}$.

Definition 1.11. If $\mathcal{L}$ is a $[0,1]$-valued logic, φ is an S-sentence of $\mathcal{L}$ and M is an S-structure such that $\varphi^M = 1$, we say that M *satisfies* φ, or that M is a *model* of φ, and write $M \models_{\mathcal{L}} \varphi$.

Note that if $\mathcal{L}$ is $[0,1]$-valued logic, then $(\mathrm{Sent}_{\mathcal{L}}, \models_{\mathcal{L}})$ is a logic in the sense of Definition 1.3. Therefore we may apply to $\mathcal{L}$ all the concepts and properties defined so far for plain logics.

Definition 1.12. The *Łukasiewicz implication* is the function $\to_{\mathrm{L}}$ from $[0,1]^2$ into $[0,1]$ defined by
$$x \to_{\mathrm{L}} y = \min\{1 - x + y, 1\}.$$

Note that $x \to_{\mathrm{L}} y = 1$ is and only if $x \leq y$.

Definition 1.13. We will say that a $[0,1]$-valued logic $\mathcal{L}$ is *closed under the basic connectives* if the following conditions hold for every vocabulary S:
  (1) If $\varphi, \psi \in \mathrm{Sent}_{\mathcal{L}}(S)$, then there exists a sentence $\varphi \to_{\mathrm{L}} \psi \in \mathrm{Sent}_{\mathcal{L}}(S)$ such that $(\varphi \to_{\mathrm{L}} \psi)^M = (\varphi)^M \to_{\mathrm{L}} (\psi)^M$ for every S-structure M.
  (2) For each rational $r \in [0,1]$, the set $\mathrm{Sent}_{\mathcal{L}}(S)$ contains a sentence with constant truth value r. These sentences are called the *constants* of $\mathcal{L}$.

Notation 1.14. If $\mathcal{L}$ is a $[0,1]$-valued logic that is closed under the basic connectives φ is a sentence of $\mathcal{L}$, and r is a constant of $\mathcal{L}$, we will write $\varphi \leq r$ and $\varphi \geq r$, as abbreviations, respectively, of $\varphi \to_{\mathrm{L}} r$ and $r \to_{\mathrm{L}} \varphi$.

Remark 1.15. Let $\mathcal{L}$ be a $[0,1]$-valued logic and let S be a vocabulary. If M is an S-structure of $\mathcal{L}$, φ is an S-sentence of $\mathcal{L}$, and r is a constant of $\mathcal{L}$, then $M \models_{\mathcal{L}} \varphi \leq r$ if and only if $\varphi^M \leq r$, and $M \models_{\mathcal{L}} \varphi \geq r$ if and only if $\varphi^M \geq r$; thus, the truth value φ^M is determined by either of the sets
$$\{\, r \in \mathbb{Q} \cap [0,1] \mid M \models_{\mathcal{L}} \varphi \leq r \,\}, \qquad \{\, r \in \mathbb{Q} \cap [0,1] \mid M \models_{\mathcal{L}} \varphi \geq r \,\}.$$

The following proposition will be invoked multiple times in the paper; the proof is left to the reader.

Proposition 1.16. *Let $\mathcal{L}$ be a $[0,1]$-valued logic that is closed under the basic connectives, let φ be an S-sentence of $\mathcal{L}$, and let r, s be constants of $\mathcal{L}$. Then, for every S-structure M, one has*
  (1) $\varphi^M \leq (\varphi \geq r)^M$.
  (2) $((\varphi \geq r) \geq s)^M = (\varphi \geq (r + s - 1))^M$.

Notation 1.17. If $\mathcal{L}$ is a $[0,1]$-valued logic that is closed under the basic connectives and φ, ψ are sentences of $\mathcal{L}$, we write $\neg\varphi$ and $\varphi \vee \psi$, as abbreviations, respectively, of $\varphi \to_{\mathrm{L}} 0$ and $(\varphi \to_{\mathrm{L}} \psi) \to_{\mathrm{L}} \psi$, and $\varphi \wedge \psi$ as an abbreviation of $\neg(\neg\varphi \vee \neg\psi)$.

Note that for every S-structure M, one has
$$\begin{aligned}
(\varphi \leq 0)^M &= 1 - (\varphi)^M, \\
(\varphi \vee \psi)^M &= \max\{\varphi^M, \psi^M\}, \\
(\varphi \wedge \psi)^M &= \min\{\varphi^M, \psi^M\}.
\end{aligned}$$

In particular, every $[0,1]$-valued logic that is closed under the basic connectives is closed under conjunctions and disjunctions.

We will refer to any function from $[0,1]^n$ into $[0,1]$, where n is a nonnegative integer, as am n-ary *connective*. The Łukasiewicz implication and the Pavelka

constants are continuous connectives, as are all the projections $(x_1, \ldots, x_n) \mapsto x_i$. The following proposition states that any other other continuous connective can be approximated by finite combinations of these.

Proposition 1.18. *Let $\mathcal{C}$ be the class of connectives generated by the Łukasiewicz implication, the Pavelka constants, and the projections through composition. Then every continuous connective is a uniform limit of connectives in $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ is closed under the connectives $\max\{x, y\}$ and $\min\{x, y\}$, by the Stone-Weierstrass theorem for lattices [GJ76, pp. 241-242], we only need to show that the connectives rx, where r is a dyadic rational, can be approximated by connectives in $\mathcal{C}$.

Notice that if $x \in [0, 1]$,

$$\frac{1}{2}x = \lim_n \bigvee_{i=1}^{n} \left(\frac{i}{n} \wedge \neg(x \rightarrow_{\mathrm{L}} \frac{i}{n}) \right).$$

Hence, since the truncated sum $\oplus : [0,1]^2 \to [0,1]$ is in $\mathcal{C}$ (as $x \oplus y = \neg x \rightarrow_{\mathrm{L}} y$), so are all the connectives $(\frac{1}{2}x + \cdots + \frac{1}{2^n})x$, for any positive integer n. $\square$

The following concept will be invoked in the statement of the Main Theorem (Theorem 4.1).

Definition 1.19. Let $\mathcal{L}$ be a $[0, 1]$-valued logic. We say that $\mathcal{L}$ is *closed under existential quantifiers* if given any any S-formula $\varphi(x)$ there exists an S-formula $\exists x \varphi$ such that for every S-structure M one has $(\exists x \varphi)^M = \sup_{a \in M}(\varphi[a]^M)$. Similarly, we say that $\mathcal{L}$ is *closed under universal quantifiers* if given any any S-formula $\varphi(x)$ there exists an S-formula $\forall x \varphi$ such that for every S-structure M one has $(\forall x \varphi)^M = \inf_{a \in M}(\varphi[a]^M)$.

1.5. **Basic continuous logic, continuous logic, and Łukasiewicz-Pavelka logic.** In this subsection we define two particular $[0, 1]$-valued logics that have been studied in the literature, namely, the continuous logic framework of [BYU10] and the Łukasiewicz-Pavelka logic (see Section 5.4 of [Háj98]). Both are logics for continuous metric structures. Traditionally, both logics have focused on particular classes of structures: the emphasis in continuous logic is in complete structures with arbitrary uniform continuity moduli, while in Łukasiewicz-Pavelka logic the focus has been on structures whose operations and predicates are 1-Lipschitz. However, model-theoretically, both can be seen as restrictions to specific classes of structures of a logic $\mathcal{L}_0$ that we introduce here and call *basic continuous logic*. More precisely, we will define two base logics: $\mathcal{L}_0$ and an extension $\mathcal{L}_1$ of $\mathcal{L}_0$; the logic $\mathcal{L}_1$ has a richer syntax than $\mathcal{L}_0$, but, as we observe the two logics are equivalent for model-theoretic purposes (see Remark 1.20 and Proposition 2.6). Continuous logic is the restriction of $\mathcal{L}_1$ to the class of complete structures, and Łukasiewicz-Pavelka logic is the restriction of $\mathcal{L}_0$ to the class of 1-Lipschitz structures.

Basic continuous logic. The class of structures of basic continuous logic, $\mathcal{L}_0$, is the class of structures is the class of all metric structures. The class of sentences of $\mathcal{L}_0$ is defined as follows. For a vocabulary S, the concept of S-term is defined as in first-order logic. If $t(x_1, \ldots, x_n)$ is an S-term (where $x_1, \ldots, x_n$ are the variables that occur in t), M is an S-structure, and $a_1, \ldots, a_n$ are elements of M, the interpretation $t^M[a_1, \ldots, a_n]$ is defined as in first-order logic as well. The atomic formulas of S are all the expressions of the form $d(t_1, t_2)$ or $R(t_1, \ldots, t_n)$,

where R is an n-ary predicate symbol of S. If $\varphi(x_1, \ldots, x_n)$ is an atomic S-formula with variables $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ are elements of and S-structure M, the interpretation $\varphi^M[a_1, \ldots, a_n]$ is defined naturally by letting

$$R(t_1, \ldots, t_n)^M[a_1, \ldots, a_n] = R^M(t_1^M[a_1, \ldots, a_n], \ldots, t_n^M[a_1, \ldots, a_n])$$

and

$$d(t_1, t_2)^M[a_1, \ldots, a_n] = d^M(t_1^M[a_1, \ldots, a_n], t_2^M[a_1, \ldots, a_n]).$$

The S-formulas of basic continuous logic are the syntactic expressions that result from closing the atomic formulas of S under the Łukasiewicz implication, the Pavelka constants, and the existential quantifier; formally, the concept of S-formula and the interpretation φ^M of a formula φ in a given structure M are defined inductively by the following rules:

- All atomic formulas of S are S-formulas.
- If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are S-formulas, then $\psi \to_{\mathrm{L}} \psi$ is a formula; if $a_1, \ldots, a_n \in M$, the interpretation $(\psi \to_{\mathrm{L}} \psi)^M[a_1, \ldots, a_n]$ is defined as $(\psi^M[a_1, \ldots, a_n]) \to_{\mathrm{L}} (\psi^M[a_1, \ldots, a_n])$.
- If $\varphi(x_1, \ldots, x_n, x)$ is an S-formula, then $\exists x \varphi$ is a formula; if $a_1, \ldots, a_n \in M$, the interpretation $(\exists x \varphi)^M[a_1, \ldots, a_n]$ is defined as $\sup_{a \in M}(\varphi[a]^M[a_1, \ldots, a_n])$,
- For every rational $r \in [0, 1]$ there is an S-formula, denoted also r, whose interpretation in any structure is the rational r.

A *sentence* of $\mathcal{L}_0$ is a formula without free variables, and the *truth value* of a S-sentence φ in an S-structure M is φ^M. We write $M \models_{\mathcal{L}_0} \varphi[a_1, \ldots, a_n]$ if $\varphi[a_1, \ldots, a_n]^M = 1$.

Recall that in any $[0, 1]$-valued logic that is closed under the basic connectives, the expressions $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \leq r$, and $\varphi \geq r$ are written as abbreviations of $\varphi \to_{\mathrm{L}} 0$, $(\varphi \to_{\mathrm{L}} \psi) \to_{\mathrm{L}} \psi$, $\neg(\neg\varphi \vee \neg\psi)$, $\varphi \to_{\mathrm{L}} r$, and $r \to_{\mathrm{L}} \varphi$, respectively. In basic continuous logic we also regard $\forall x \varphi$ as an abbreviation of $\neg\exists x \neg\varphi$.

The logic $\mathcal{L}_1$. The logic $\mathcal{L}_1$ is defined just as $\mathcal{L}_0$, with the difference that instead of taking the closure under the Łukasiewicz-Pavelka connectives one takes the closure under all continuous connectives (and the existential quantifier).

Proposition 1.18 gives us the following remark, which will allow us to transfer model-theoretic results between $\mathcal{L}_0$ and $\mathcal{L}_1$:

Remark 1.20. For every S-formula $\varphi(\bar{x})$ of $\mathcal{L}_1$ and for every $\epsilon > 0$ there exists a formula $\psi(\bar{x})$ of basic continuous logic such that $|\varphi^M[\bar{a}] - \psi^M[\bar{a}]| < \epsilon$ for every complete S-structure M and every tuple $\bar{a}$ in M with $\ell(\bar{a}) = \ell(\bar{x})$. It follows that if M, N are S-structures, then $M \equiv_{\mathcal{L}_0} N$ if and only if $M \equiv_{\mathcal{L}_1} N$, and $M \preceq_{\mathcal{L}_0} N$ if and only if $M \preceq_{\mathcal{L}_1} N$. Moreover, Corollary 1.23 below gives us that every structure is equivalent in $\mathcal{L}_0$ (and $\mathcal{L}_1$) to its metric completion.

Continuous logic. Continuous logic is simply the restriction of $\mathcal{L}_1$ to the class of complete metric structures. This logic was introduced by Ben Yaacov and Usvyatsov [BYU10] as a reformulation of Henson's logic for metric spaces; Ben Yaacov and Usvyatsov called it *continuous first-order-logic*. [1]

---

[1]Continuous logic is an instance of the more general concept of continuous logic studied by Chang and Keisler in [CK66], and was proposed by Ben Yaacov and Usvyatsov as a reformulation of Henson's model theory of complete metric spaces. For a survey of Henson's logic as it regards structures based on normed spaces, see [HI02]. The more general framework devised by Henson for arbitrary continuous metric structures was never published.

Łukasiewicz-Pavelka logic. The formulas of Łukasiewicz-Pavelka logic are like those of basic continuous logic, with the difference that in place of the distinguished metric d one uses the *similarity* relation $x \approx y$. However, there is a precise correspondence between the two relations, namely, $d(x, y)$ is $1 - (x \approx y)$ (in other words, the two relations are negations of each other); see Section 5.6 of [Háj98], especially Example 5.6.3-(1). Also, in Łukasiewicz-Pavelka logic, in place of the uniform continuity requirement given in Definition 1.2, for each n-ary operation symbol f, one has the axiom

$$(x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n) \to_{\mathrm{L}} (f(x_1 \ldots, x_n) \approx f(y_1, \ldots y_n)),$$

and similarly, for each n-ary predicate symbol R, one has the axiom[2]

$$(x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n) \to_{\mathrm{L}} (R(x_1 \ldots, x_n) \leftrightarrow_L R(y_1, \ldots y_n)).$$

See Definition 5.6.5 of [Háj98]. Thus, Łukasiewicz-Pavelka logic is the restriction of basic continuous logic to the class of 1-Lipschitz structures, i.e., structures whose operations and predicates are 1-Lipschitz.

Historically, Pavelka extended Łukasiewicz propositional logic by adding the rational constants, and proved a form of approximate completeness for the resulting logic. See [Pav79a, Pav79b, Pav79c] (see also Section 5.4 of [Háj98].) This is known as Pavelka-style completeness. Łukasiewicz-Pavelka logic is also referred to in the literature as rational Pavelka logic, or Pavelka many-valued logic. Novák proved Pavelka-style completeness for predicate Łukasiewicz-Pavelka logic, which he calls "first-order fuzzy logic", first using ultrafilters [Nov89, Nov90], and later using a Henkin-type construction [Nov95]. Another proof of Pavelka-style completeness for predicate Łukasiewicz-Pavelka logic was given by Hajek; see [Háj97] and [Háj98, Section 5.4]. Hajek, Paris, and Stepherson have proved that Łukasiewicz-Pavelka logic is a conservative extension of Łukasiewicz predicate logic [HPS00].

Model-theoretic properties of $\mathcal{L}_0$ and $\mathcal{L}_1$. We now state, without proof, some of the fundamental properties of basic continuous logic. Versions of Theorems 1.21, 1.22, and 1.25, were first proved by Henson in the mid 1970's, for Banach spaces instead of general metric structures, and using Henson's logical formalism of positive bounded formulas instead of $[0, 1]$-valued logics. (Henson's apparatus was one of the main motivations for the development of continuous logic.)

The most distinctive model-theoretic property of basic continuous logic is compactness:

Theorem 1.21 (Compactness of basic continuous logic). *Let $\mathbf{S}$ be a signature and let T be an S-theory of basic continuous logic. If every finite subtheory of T is satisfied by an $\mathbf{S}$-structure, then T is satisfied by an $\mathbf{S}$-structure; furthermore, this structure can be taken to be complete.*

Theorem 1.21 can be proved by taking ultraproducts of $[0, 1]$-valued structures. The argument is simple, but we omit the details, as they are not directly relevant to the rest of the paper.

One obtains compactness for continuous logic by restricting Theorem 1.21 to complete structures and invoking Remark 1.20. Restricting the theorem to 1-Lipschitz structures yields compactness of Łukasiewicz-Pavelka logic, but it must be noted that in this case the 1-Lipschitz condition makes it unnecessary to fix uniform

---

[2]Here we use $(\alpha \leftrightarrow_L \beta)$ as an abbreviation of $(\alpha \to_{\mathrm{L}} \beta) \wedge (\beta \to_{\mathrm{L}} \alpha)$.

continuity moduli; in other words, for Łukasiewicz-Pavelka logic, the statement of Theorem 1.21 holds with the signature **S** replaced by a vocabulary S.

If M is a structure and A is a subset of the universe of M, we denote by $\langle A \rangle$ the closure of A under the functions of M, and by $M \upharpoonright \langle A \rangle$ the substructure of M induced by $\langle A \rangle$.

Theorem 1.22 (Tarski-Vaught test for basic continuous logic). *Let M be an S-structure. If A is a subset of M and $S(A)$ is extension of S that includes a constant symbol for each element of A, the following conditions are equivalent:*

(1) *$M \upharpoonright \langle A \rangle \preceq_{\mathcal{L}_0} M$.*
(2) *For every $S(A)$-formula $\varphi(x)$, if $M \models_{\mathcal{L}_0} \varphi[b]$ for some element b of M, then for every rational $r \in (0,1)$ there is an S-term $t(\bar{x})$ and elements $\bar{a}$ of A such that $M \models_{\mathcal{L}_0} \varphi[t(\bar{a})] \geq r$.*

A direct consequence of Theorem 1.22 is the following property:

Corollary 1.23. *Let M be a metric structure and let N be a substructure of M.*

(1) *If N is dense in M, then $N \preceq_{\mathcal{L}_0} M$. In particular, every structure is equivalent in $\mathcal{L}_0$ to its metric completion.*
(2) *If $N \preceq_{\mathcal{L}_0} M$, and $\overline{N}^M$ is the topological closure of N in M, then $\overline{N}^M \preceq_{\mathcal{L}_0} M$*

It follows from the main result of [Iov01] that no $[0,1]$-valued logic for continuous metric structures that extends basic continuous logic properly satisfies the compactness property and the Tarski-Vaught test. In other words, basic continuous logic is maximal with respect to these two properties.

Now we turn our attention to the downward Löwenheim-Skolem-Tarski theorem for basic continuous logic. We will use two versions of this theorem; the first version is for arbitrary continuous metric structures (Theorem 1.24) and the second one is for complete structures (Theorem 1.25). In the continuous logic literature, where complete structures are emphasized, only the second version is stated; however, the standard argument used to prove the second version proceeds by first showing that the first version holds, and then taking completions and invoking Corollary 1.23. We state the first version as a separate theorem, for it will be needed in the proof of Lemma 3.15).

Theorem 1.24 (Löwenheim-Skolem-Tarski theorem for continuous metric structures). *For every S-structure M and every subset A of M there exists a substructure N of M such that*

(1) *$N \preceq_{\mathcal{L}_0} M$,*
(2) *$A \subseteq N$,*
(3) *$|N| \leq |A| + |S| + \aleph_0$.*

The version for complete metric structures is analogous, but in this context the "correct" measure of size of a structure is its density, rather than its cardinality. The *density character* (or simply *density*) of a metric space M, denoted $\text{density}(M)$, is the smallest cardinal of a dense subset of M.

Theorem 1.25 (Löwenheim-Skolem-Tarski theorem for complete metric structures). *For every complete S-structure M and every subset A of M there exists a complete substructure N of M such that*

(1) $N \preceq_{\mathcal{L}_0} M$,
(2) $A \subseteq N$,
(3) $\mathrm{density}(N) \leq \mathrm{density}(A) + |S| + \aleph_0$.

1.6. **Relativizations to discrete predicates.** It is often helpful to know that the fact that a predicate is discrete, in the sense that it only takes on values in $\{0, 1\}$, can be expressed using formulas of basic continuous logic:

Definition 1.26. Let M be an S-structure and let P a predicate symbol of S. We define $\mathrm{Discrete}\big(P(\bar{x})\big)$ to be the S-formula

$$P(\bar{x}) \vee \neg P(\bar{x}),$$

and call P^M *discrete* if $M \models \forall \bar{x} \ \mathrm{Discrete}\big(P(\bar{x})\big)$.

Definition 1.26 will play an important role in the proof of the Main Theorem (Theorem 4.1).

Let S be a vocabulary and let $P(x)$ be a monadic predicate not in S. If M is a $(S \cup \{P\})$-structure such that P^M is discrete, and a valid S-structure of $\mathcal{L}$ is obtained by restricting the universe of M to $\{a \in M \mid M \models_{\mathcal{L}} P[a]\}$, we denote this structure by $M \upharpoonright \{x \mid P(x)\}$. Note that if M is complete, the continuity of P ensures that the preceding structure, when defined, is complete. If φ is an S-formula of continuous logic, the *relativization* of φ to P, denoted $\varphi^{\{x|P(x)\}}$ or φ^P, is the $(S \cup \{P\})$-formula defined by the following recursive rule:

- If φ is atomic, then $\varphi^{\{x|P(x)\}}$ is φ.
- If φ is of the form $C(\psi_1, \ldots, \psi_n)$, where C is a connective, then $\varphi^{\{x|P(x)\}}$ is $C(\psi_1^{\{x|P(x)\}}, \ldots, \psi_n^{\{x|P(x)\}})$.
- If φ is of the form $\exists x \psi$, then $\varphi^{\{x|P(x)\}}$ is $\exists y(\psi(y) \wedge \psi^{\{x|P(x)\}})$.
- If φ is of the form $\forall x \psi$, then $\varphi^{\{x|P(x)\}}$ is $\forall y(\neg \psi(y) \vee \psi^{\{x|P(x)\}})$.

Then, if M and P are as above, we have

$$(\varphi^{\{x|P(x)\}})^M = \varphi^{M \upharpoonright \{x|P(x)\}}.$$

In the preceding rule, instead of the monadic predicate $P(x)$ we may have a binary predicate symbol $R(x, y)$; this shows that the logics discussed in this section have the following property:

Definition 1.27. We will say that a $[0, 1]$-valued logic $\mathcal{L}$ permits *relativization to definable families of predicates* if for every vocabulary S, every S-sentence φ and every predicate symbol $R(x, y)$ not in S there is an $(S \cup \{R\})$-formula $\psi(x)$, denoted $\varphi^{\{y|R(x,y)\}}(x)$, such that the following holds: whenever M is an $(S \cup \{R\})$-structure such that for every a in M,

- either $M \models R[a, b]$ or $M \models \neg R[a, b]$ for every b in M, and
- $(M, a) \upharpoonright \{y \mid R(x, y)\}$ is defined as a structure of $\mathcal{L}$,

one has

$$(\varphi^{\{y|R(x,y)\}})^M[a] = \varphi^{M \upharpoonright \{y|R(x,y)\}}[a].$$

In this case, we call the formula $\varphi^{\{y|R(x,y)\}}(x)$ a *relativization* of φ to $\{y \mid R(x, y)\}$.

A logic $\mathcal{L}$ has the Löwenheim-Skolem property for sentences if every sentence of $\mathcal{L}$ that has a model has a countable model. The first author has proved the following version of Lindström's first theorem [Cai1]:

Theorem 1.28. *Let $\mathcal{L}$ be an extension of continuous logic that satisfies the following properties:*

> · *Closure under the Łukasiewicz-Pavelka connectives and relativization to discrete predicates,*
> · *Compactness,*
> · *The Löwenheim-Skolem property for sentences.*

Then $\mathcal{L}$ is equivalent to continuous logic.

A similar result holds for extensions of first order Łukasiewicz-Pavelka logic, under the additional assumption that any consistent theory has a complete model.

1.7. **Beyond the interval** $[0, 1]$**.** For simplicity, we have focused our attention on continuous metric structures of diameter bounded by 1. The formalism of basic continuous logic can be adapted to cover bounded structures of arbitrary diameter. However, the resulting logic is not compact, but rather locally compact, in the following sense:

Theorem 1.29. *Let $\mathbf{S}$ be a signature. For every bounded $\mathbf{S}$-structure M there is an S-sentence φ and a rational $r \in (0, 1)$ such that $M \models \varphi$ and the following property holds. If T is an S-theory such that every finite subset of $T \cup \{\varphi \geq r\}$ is satisfied by an $\mathbf{S}$-structure, then $T \cup \{\varphi \geq r\}$ is satisfied by an $\mathbf{S}$-structure.*

The main results of the paper, namely, Theorems 3.6 and 4.1 hold under this wider semantics, although the second result holds locally only. One way to obtain this generalization is to use 0 instead of 1 as designated truth value for $\models$, and the truncated subtraction on $[0, 1]$ instead of the Łukasiewicz implication.

2. Logics and Topologies

In this section we associate with every logic $\mathcal{L}$ a topology that we call the *logic topology* of $\mathcal{L}$. The idea of using the logic topology to study properties of abstract logics is due to the first author [Cai93, Cai95, Cai99]. We will focus on logics whose logic topology is regular.

Let $\mathbf{S} = (S, \mathcal{U})$ be a signature. If $\mathcal{L}$ is a logic, we will denote by $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$ the class of $\mathbf{S}$-structures of $\mathcal{L}$. For every S-theory T of $\mathcal{L}$, define

$$\mathrm{Mod}_{\mathbf{S}}(T) = \{\, M \in \mathrm{Str}_{\mathcal{L}}(\mathbf{S}) \mid M \models_{\mathcal{L}} T \,\}.$$

The following is the main definition of this section. This concept will play a central role in our arguments.

Definition 2.1. Let $\mathcal{L}$ be a logic and let $\mathbf{S} = (S, \mathcal{U})$ be a signature. The *logic topology* on $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$, denoted $\tau_{\mathcal{L}}(\mathbf{S})$, is the topology on $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$ whose closed classes are those of $\mathrm{Mod}_{\mathbf{S}}(T)$, where T is a theory.

The following proposition follows directly from the definitions.

Proposition 2.2. *Let $\mathcal{L}$ be a logic and let $\mathbf{S} = (S, \mathcal{U})$ be a signature.*

> (1) *$\mathcal{L}$ is compact if and only if the space $(\mathrm{Str}_{\mathcal{L}}(\mathbf{S}), \tau_{\mathcal{L}}(\mathbf{S}))$ is compact, for every signature $\mathbf{S}$.*
> (2) *$\mathcal{L}$ is λ-compact if and only if the space $(\mathrm{Str}_{\mathcal{L}}(\mathbf{S}), \tau_{\mathcal{L}}(\mathbf{S}))$ is λ-compact, for every signature $\mathbf{S}$.*

Since we are assuming that all logics are closed under finite disjunctions (see Convention 1.4), the classes of the form

$$\mathrm{Mod}_{\mathbf{S}}(\varphi) = \{ M \in \mathrm{Str}_{\mathcal{L}}(\mathbf{S}) \mid M \models_{\mathcal{L}} \varphi \},$$

where $\varphi \in \mathrm{Sent}_{\mathcal{L}}(S)$, are closed under finite unions. These classes form a base of closed classes for the logic topology on $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$.

Convention 2.3. Let $\mathcal{L}$ be a $[0,1]$-valued logic that is closed under the basic connectives and let $\mathbf{S}$ be a signature. If φ is a sentence of $\mathcal{L}$, and $r \in [0,1]$ is a constant of $\mathcal{L}$, we write

$$\mathrm{Mod}_{\mathbf{S}}(\varphi < r) \qquad \text{and} \qquad \mathrm{Mod}_{\mathbf{S}}(\varphi > r),$$

respectively, as abbreviations for the classes

$$\mathrm{Str}_{\mathcal{L}}(\mathbf{S}) \setminus \mathrm{Mod}_{\mathbf{S}}(\varphi \geq r) \qquad \text{and} \qquad \mathrm{Str}_{\mathcal{L}}(\mathbf{S}) \setminus \mathrm{Mod}_{\mathbf{S}}(\varphi \leq r).$$

Note that the classes of the form $\mathrm{Mod}_{\mathbf{S}}(\varphi < 1)$ where $\varphi \in \mathrm{Sent}_{\mathcal{L}}(S)$ for a base for $\tau_{\mathcal{L}}(\mathbf{S})$. By replacing φ with $\neg\varphi$, it follows that the classes of the form $\mathrm{Mod}_{\mathbf{S}}(\varphi < 1)$ for a base for $\tau_{\mathcal{L}}(\mathbf{S})$ as well.

2.1. **Comparing logics through their topologies.** Recall that a topological space X is regular if and only it has a local base of closed neighborhoods, i.e., whenever $x \in X$ and U is a neighborhood of x there exists a neighborhood W of x such that $\overline{W} \subseteq U$. We shall focus our attention on logics whose logic topology is regular. The following observation is useful to compare logics.

If (X,τ) is topological space and $x,y \in X$, one says that x and y are τ-*indistinguishable*, denoted $x \overset{\tau}{\equiv} y$, if every τ-neighborhood of x contains y and every τ-neighborhood of y contains x. Note that if (X,τ) is a regular topological space and $x,y \in X$, then $x \overset{\tau}{\equiv} y$ if and only if every τ-neighborhood of x contains y. Also, $\{ z \in X \mid z \overset{\tau}{\equiv} x \}$ is the τ-closure of $\{x\}$. If τ, τ' are topologies on X and any two τ-indistinguishable points of X are also τ'-indistinguishable, we will write $\overset{\tau}{\equiv} \Rightarrow \overset{\tau'}{\equiv}$.

Suppose that τ and τ' are topologies on a set X such that $\tau \subseteq \tau'$. Clearly, $\overset{\tau'}{\equiv} \Rightarrow \overset{\tau}{\equiv}$. The following proposition shows that under sufficient compactness of τ', from $\overset{\tau}{\equiv} \Rightarrow \overset{\tau'}{\equiv}$ one can obtain $\tau = \tau'$.

Recall that the *weight* of a topological space (X,τ) is the smallest possible cardinality of a base for τ.

Proposition 2.4. *Let τ, τ' be regular topologies on X such that $\tau \subseteq \tau'$ and $\overset{\tau}{\equiv} \Rightarrow \overset{\tau'}{\equiv}$. If τ has weight λ and and τ' is λ-compact, then $\tau' = \tau$.*

Proof. Let $\mathcal{B}$ be a base for τ of cardinality λ, and fix $U \in \tau'$ in order to prove $U \in \tau$. Fix now $x \in U$ and $y \in U^c$. Since x and y are τ'-topologically distinguishable and $\overset{\tau}{\equiv} \Rightarrow \overset{\tau'}{\equiv}$, there exist disjoint τ-open sets $V_{x,y}, W_{x,y}$ such that $x \in V_{x,y}$ and $y \in W_{x,y}$. Without loss of generality, we may assume $W_{x,y} \in \mathcal{B}$. Since $W_{x,y} \in \tau \subseteq \tau'$, allowing y to range over all elements of U^c and using the λ-compactness of τ', we obtain $V_x \in \tau$ such that $x \in V_x$ and $V_x \subseteq U$. Since x is arbitrary in U, this shows that U is τ-open. $\qquad\square$

If $\mathbf{S}$ is a signature and M, N are $\mathbf{S}$-structures of a logic $\mathcal{L}$, then, clearly, $M \equiv_{\mathcal{L}} N$ if and only if M and N are indistinguishable in the logic topology $\tau_{\mathcal{L}}(\mathbf{S})$ (see Definition 2.1). If $\mathcal{L}$ and $\mathcal{L}'$ are logics with the same class of structures, we will write $\equiv_{\mathcal{L}} \Rightarrow \equiv_{\mathcal{L}'}$ if $\overset{\tau_{\mathcal{L}}(\mathbf{S})}{\equiv} \Rightarrow \overset{\tau_{\mathcal{L}'}(\mathbf{S})}{\equiv}$ for every signature $\mathbf{S}$.

Definition 2.5. Let $\mathcal{L}, \mathcal{L}'$ be logics with the same class of structures. We will say that $\mathcal{L}'$ *extends* $\mathcal{L}$, and write $\mathcal{L} \prec \mathcal{L}'$, if $\tau_{\mathcal{L}}(\mathbf{S}) \subseteq \tau_{\mathcal{L}'}(\mathbf{S})$ for every signature $\mathbf{S}$. We will say that $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent*, and write $\mathcal{L} \sim \mathcal{L}'$, if both $\mathcal{L} \prec \mathcal{L}'$ and $\mathcal{L}' \prec \mathcal{L}$ hold.

Note that the relation $\prec$ defined above compares logics with respect to their theories, and not their sentences.

Recall from Section 1.5 that $\mathcal{L}_0$ denotes continuous logic and $\mathcal{L}_1$ denotes the extension of $\mathcal{L}_0$ that results from adding all continuous connectives.

Proposition 2.6. $\mathcal{L}_0 \sim \mathcal{L}_1$.

Proof. Immediate from Remark 1.20. $\qquad\square$

Proposition 2.7. *Let $\mathcal{L}$ be a $[0,1]$-valued logic that is closed under the basic connectives and let $\mathbf{S} = (S, \mathcal{U})$ be a signature.*

(1) *Every S-sentence is a continuous function from $(\mathrm{Str}_{\mathcal{L}}(\mathbf{S}), \tau_{\mathcal{L}}(\mathbf{S}))$ into $[0,1]$.*
(2) *$(\mathrm{Str}_{\mathcal{L}}(\mathbf{S}), \tau_{\mathcal{L}}(\mathbf{S}))$ is a regular topological space.*
(3) *If $\mathcal{L}$ is compact and $\mathcal{L}' \prec \mathcal{L}$, then every sentence in $\mathrm{Sent}_{\mathcal{L}'}(\mathbf{S})$ is a uniform limit in $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$ of sentences in $\mathcal{L}$.*

Proof. The first two conclusions follow readily from the observation that if φ is a sentence and r, s are rationals with $0 \leq r < s \leq 1$, then $\varphi^{-1}[r, s] = \mathrm{Mod}_{\mathbf{S}}(\varphi \geq r \wedge \varphi \leq s)$.

To prove the third one, first notice that any S-sentence of $\mathcal{L}'$ is continuous with respect to $\tau_{\mathcal{L}'}(\mathbf{S})$ and hence with respect to $\tau_{\mathcal{L}}(\mathbf{S})$, so, by continuity, it factors through the compact space $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})/\equiv_{\mathcal{L}}$. Regularity (part (2)) ensures that this space is Hausdorff. The conclusion thus follows from the Stone-Weierstrass theorem for lattices. $\qquad\square$

Remark 2.8. One can define $\mathrm{Str}_{\mathcal{L}}(S)$ and $\tau_{\mathcal{L}}(S)$ for any vocabulary S by replacing $\mathbf{S}$ with S in the definitions of $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$ and $\tau_{\mathcal{L}}(\mathbf{S})$, respectively. Then clearly, the proof (1) and (2) of Proposition 2.7 holds true with $\mathbf{S}$ replaced by S, and the result for signatures follows by restricting from $\mathrm{Str}_{\mathcal{L}}(S)$ to the subspace $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$.

3. A General omitting types theorem

The main result of this section is Theorem 3.6.

3.1. **A general version of the Baire category theorem.** Let X be a topological space and let $x \in X$ be given. Recall that a subclass G of X is a *neighborhood of x* if there is an open subclass O of G containing x. If λ is any infinite cardinal, we will say that a subclass G of X is a *λ-neighborhood of x* if G is an intersection of less than λ neighborhoods of x. We will say that G is a *λ-neighborhood* if G is a λ-neighborhood of some point in X. This definition of λ-neighborhood yields naturally definitions concepts of λ-open, λ-interior, λ-dense, etc.

Definition 3.1. Let X be a topological space and λ an infinite cardinal. We will say that X has the λ-*Baire Property* if whenever $\{D_i\}_{i<\lambda}$ is a collection of subclasses of X that are λ-open and λ-dense, the intersection $\bigcap_{i<\lambda} D_i$ is λ-dense.

Note that the ω-Baire Property is simply the classical Baire Property. Also, a topological space X has the λ-Baire Property if and only if a union of $\leq \lambda$ many λ-closed classes with empty λ-interior has empty λ-interior. This is immediate by taking complements in Definition 3.1.

Note that if X is a regular topological space and U is a λ-neighborhood of x, then there exists a λ-neighborhood W of x such that $\overline{W} \subseteq U$.

Proposition 3.2. *Every locally compact regular topological space has the λ-Baire property for every infinite cardinal λ.*

Proof. Suppose $\{D_i\}_{i<\lambda}$ are λ-open and λ-dense in X, and let O be a nonempty λ-open subclass of X. We must show that $O \cap \bigcap_{i<\lambda} D_i$ is nonempty. Below we define, recursively, a decreasing sequence $(U_i)_{i<\lambda}$ of λ-neighborhoods such that

(1) $\overline{U}_0$ is compact and $\overline{U}_0 \subseteq O$,
(2) $\overline{U}_i \subseteq D_i$, for $i < \lambda$,
(3) $\overline{U}_j \subseteq U_i$ for $i < j < \lambda$.

The construction is as follows. For $i = 0$, the set $D_0 \cap O$ is nonempty by hypothesis and λ-open by construction, so, by regularity and local compactness, there exists a λ-neighborhood U_0 such that $\overline{U}_0 \subseteq D_0 \cap O$ and $\overline{U}_0$ is compact.

Assume that $i < \lambda$ is positive and U_j has been defined for $j < i$ with the properties indicated above. Then $\bigcap_{j<i} U_j \neq \emptyset$: if i is successor, say, $i = j_0 + 1$, this is because $\bigcap_{j<i} U_j = U_{j_0}$, and if i is limit, $\bigcap_{j<i} U_j = \bigcap_{j<i} \overline{U}_j$, which is nonempty by the compactness of $\overline{U}_0$. Since D_i is λ-dense, $D_i \cap \bigcap_{j<i} U_j$ is nonempty, and by regularity of the topology we can find a λ-neighborhood U_i such that $\overline{U}_i \subseteq D_i \bigcap_{j<i} U_j$. This concludes the recursive definition. Finally, by compactness, we obtain $\emptyset \neq \bigcap_{i<\lambda} \overline{U}_i \subseteq \bigcap_{i<\lambda} D_i \cap O$.

$\square$

Corollary 3.3. *Let λ be an infinite cardinal. If X is locally compact regular topological space and $\{G_i\}_{i<\lambda}$ is a collection of λ-open subclasses of X, then $\bigcap_{i<\lambda} G_i$ has the λ-Baire property.*

Proof. Similar to the proof of Proposition 3.2, but with the added condition that $U_i \subseteq G_i$ for every $i < \lambda$. $\square$

3.2. **Statement of the general omitting types theorem.** Let $\mathcal{L}$ be a logic and let $\mathbf{S}$ be a signature. If $\bar{x} = x_1, \ldots, x_n$ a finite list of constant symbols not in S and Σ is a set of $(S \cup \{\bar{x}\})$-sentences, we emphasize this by writing Σ as $\Sigma(\bar{x})$. If M is an S-structure and $\bar{a} = a_1, \ldots, a_n$ is a tuple of elements of M such that $M \models_{\mathcal{L}} \sigma[\bar{a}]$ for every $\sigma(\bar{x}) \in \Sigma(\bar{x})$, we say that M *satisfies* $\Sigma(\bar{x})$, and that $\bar{a}$ *realizes* $\Sigma(\bar{x})$ in M. If T is an S-theory and M is an $\mathbf{S}$-structure that satisfies $T \cup \Sigma(\bar{x})$, we say that $\Sigma(\bar{x})$ is an $\mathbf{S}$-*type* of T.

If $\Gamma(\bar{x}), \Sigma(\bar{x})$ are $\mathbf{S}$-types for a theory T, we write $T, \Gamma(\bar{x}) \models_{\mathcal{L},\mathbf{S}} \Sigma(\bar{x})$ if whenever M is an $\mathbf{S}$-structure that is a model of T, every realization of $\Gamma(\bar{x})$ in M is also a realization of $\Sigma(\bar{x})$. If the signature $\mathbf{S}$ is given by the context, we may write $T, \Gamma(\bar{x}) \models_{\mathcal{L}} \Sigma(\bar{x})$ instead of $T, \Gamma(\bar{x}) \models_{\mathcal{L},\mathbf{S}} \Sigma(\bar{x})$.

Definition 3.4. Let T be a consistent S-theory and let $\Sigma(\bar{x})$ be an **S**-type of T.

(1) If λ is an uncountable cardinal, we will say that $\Sigma(\bar{x})$ is λ-*principal* over T if there exists a set of S-formulas $\Phi(\bar{x})$ of cardinality less than λ such that
    (a) $T \cup \Phi(\bar{x})$ is satisfiable by an **S**-structure, and
    (b) $T \cup \Phi(\bar{x}) \models_{\mathcal{L},\mathbf{S}} \Sigma(\bar{x})$.
(2) We will say that $\Sigma(\bar{x})$ is ω-*principal* over T if there exist terms $t_1(\bar{y}), \ldots, t_n(\bar{y})$, where $n = \ell(\bar{x})$ and a single formula $\varphi(\bar{y})$ such that
    (a) $T \cup \{\varphi(\bar{y})\}$ is satisfiable by an **S**-structure, and
    (b) $T \cup \{\varphi(\bar{y}) \geq r\} \models_{\mathcal{L},\mathbf{S}} \Sigma(t_1(\bar{y}), \ldots, t_n(\bar{y}))$, for some $r \in \mathbb{Q} \cap (0,1)$.

In each case, we will say that $\Sigma(\bar{x})$, respectively, $\varphi(\bar{x})$, *generates* $\Sigma(\bar{x})$ over T.

If $\Sigma(\bar{x})$ is an **S**-type for T and M is a **S**-structure that is a model of T, we will say that M *omits* $\Sigma(\bar{x})$ if it does not realize it, i.e., if for every $\bar{a} \in M$, there is $\sigma(\bar{x}) \in \Sigma(\bar{x})$ such that $M \models_{\mathcal{L}} \sigma[\bar{a}] < 1$.

Definition 3.5. Let $\mathcal{L}$ be a logic and λ an infinite cardinal. We will say that $\mathcal{L}$ has the λ-*omitting types property* if whenever T is an S-theory of cardinality $\leq \lambda$ that is satisfied by an **S**-structure and $\{\Sigma_j(\bar{x})\}_{j<\lambda}$ is a set of **S**-types that are not λ-principal over T there is a model of T that omits each $\Sigma_j(\bar{x})$.

The following is the main result of this section.

Theorem 3.6. *Basic continuous logic has the λ-omitting types property for every infinite cardinal λ.*

Remarks 3.7.

(1) The reader may have noticed an asymmetry between the uncountable case and the countable case in Definition 3.4, as the latter involves terms while the former does not. The following example shows that the presence of terms in the countable case is essential for Theorem 3.6 to hold, even for complete structures. Consider the structure that consists of the closed unit interval $[0,1]$ equipped with the usual metric and constants for 0 and for every element of the form $1/n$ where n is a positive integer; let T be the complete theory of this structure and let $\Sigma(x)$ be the type of 0. Then $\Sigma(x)$ cannot be omitted by any model of T; however, there is no pair $(\varphi(x), r)$ with $r \in \mathbb{Q} \cap (0,1)$ such that $T \cup \{\varphi(x) \geq r\} \models_{\mathcal{L}_0} \Sigma(x)$ because, since $r < 1$, every model of T satisfies $\varphi(1/n)$ for sufficiently large n.

In [KM01], Keisler and Miller prove a generalization of Morley's categoricity theorem for theories without equality. For this, they replace the standard notion of model by what they call "reduced model", and their proof follows a parallel development, for reduced models, of the machinery needed for Morley's classical proof (for countable languages). The notion of principal type needed for their proof coincides with the concept of ω-principal given in Definition 3.4.

(2) A type that is omitted by a structure may not be omitted by its metric its completion (take, for example, a nonconvergent Cauchy sequence, and consider the complete type of its limit in the signature expanded with constants for the terms of the sequence). In Section 3.5 we study the question of what types can be omitted by complete structures; we introduce a concept of *metrically principal type* and, as a corollary of Theorem 3.6, we obtain a counterpart of Theorem 3.6 for metrically principal types and complete

metric structures; see Propositions 3.17. We observe that the special case of this result when $\lambda = \omega$ is Henson's omitting types theorem for complete metric structures (see Remark 3.21).

An S-theory T is *complete* if for every S-sentence φ and every $r \in \mathbb{Q} \cap (0,1)$, either $\varphi \leq r$ or $\varphi \geq r$ is in T. Note that this yields a concept of complete type, since, in our context, types are theories.

Remarks 3.8.

(1) If T is a complete theory, then all models of T satisfy the same uniform continuity moduli.
(2) If T is complete, every type that is ω-principal over T is realized in every model of T. Hence, by Theorem 3.6, for T complete, a type is ω-principal over T if and only if it is realized in every model of T.

The rest of this section falls into into two parts. In the first part, we study the link between our uncountable version of the Baire category theorem and the logic topology for regular logics. In the second part we prove Theorem 3.6.

3.3. The λ-Baire property and classes of structures.

Definition 3.9. A logic $\mathcal{L}$ is *locally compact* if the space $(\operatorname{Str}_{\mathcal{L}}(\mathbf{S}), \tau_{\mathcal{L}}(\mathbf{S}))$ is locally compact for every signature $\mathbf{S}$, and *regular* if $(\operatorname{Str}_{\mathcal{L}}(\mathbf{S}), \tau_{\mathcal{L}}(\mathbf{S}))$ is regular for every signature $\mathbf{S}$.

By Proposition 2.7-(2), every $[0,1]$-logic that is closed under the basic connectives is regular.

Proposition 3.10. *Let $\mathcal{L}$ be a locally compact regular logic and let λ be an infinite cardinal. Then,*

(1) $\operatorname{Str}_{\mathcal{L}}(\mathbf{S})$ *has the λ-Baire Property.*
(2) *If T is an S-theory, then $\operatorname{Mod}_{\mathbf{S}}(T)$ has the λ-Baire Property.*

Proof. The first part is given by Proposition 3.2. For the second part, notice that since $\operatorname{Mod}_{\mathbf{S}}(T)$ is closed, it inherits local compactness from $\operatorname{Str}_{\mathcal{L}}(\mathbf{S})$, so this part follows from Proposition 3.2 as well. $\qquad\square$

For the rest of this section, the background logic is basic continuous logic, and the background topology is the logic topology on $\operatorname{Str}_{\mathcal{L}}(\mathbf{S})$, where $\mathbf{S} = (S, \mathcal{U})$ is a fixed signature of cardinality $\leq \lambda$.

Let $C = (c_i)_{i<\lambda}$ be a family of new constants. We denote by $(\mathbf{S} \cup C)$ the signature that results from adding the constants in C to $\mathbf{S}$. We will denote $(\mathbf{S} \cup C)$-structures as $(M, \vec{a})$, where M is an $\mathbf{S}$-structure and $\vec{a} = (a_i)_{i<\lambda}$ interprets $(c_i)_{i<\lambda}$. In this context, $\langle \vec{a} \rangle$ denotes the closure of $\vec{a}$ under the functions of M, and $M \restriction \langle \vec{a} \rangle$ denotes the substructure of M induced by $\langle \vec{a} \rangle$.

For the rest of the section, $\mathcal{W}$ will denote the class of $(\mathbf{S} \cup C)$-structures of the form $(M, \vec{a})$ such that $M \restriction \langle \vec{a} \rangle$ is an elementary substructure of M.

Proposition 3.11. *Let λ be an infinite cardinal and let T be a $(S \cup C)$-theory of cardinality at most λ that is satisfied by an $\mathbf{S}$-structure. Then the class $\mathcal{W} \cap \operatorname{Mod}_{\mathbf{S}}(T)$ is nonempty and has the λ-Baire Property.*

Proof. Let $\big(\varphi_i(x)\big)_{i<\lambda}$ be a list of all the $(S \cup C)$-formulas in one variable, and $(t_i)_{i<\lambda}$ a list of all closed $(S \cup C)$-terms. By the Tarski-Vaught test (Theorem 1.22), we have $(M, \vec{a}) \in \mathcal{W}$ if and only if the following condition holds: for every $i < \lambda$ and $r \in \mathbb{Q} \cap (0,1)$, if $M \models_{\mathcal{L}_0} \exists x\, \varphi_i(x)$, then $M \models_{\mathcal{L}_0} \varphi_i(t_j) > r$ for some $j < \lambda$. This means that

$$\mathcal{W} = \bigcap_{i<\lambda} \ \bigcap_{r \in \mathbb{Q} \cap (0,1)} \left(\mathrm{Mod}_{\mathbf{S}}\big(\exists x\, \varphi_i(x) < 1\big) \cup \bigcup_{j<\lambda} \mathrm{Mod}_{\mathbf{S}}\big(\varphi_i(t_j) > r\big)\right).$$

Thus, by Corollary 3.3, the class $\mathcal{W}$ has the λ-Baire property. The class $\mathcal{W} \cap \mathrm{Mod}_{\mathbf{S}}(T)$ has the λ-Baire property by Corollary 3.3 and Proposition 3.10. This class is non empty because, by the downward Löwenheim-Skolem-Tarski theorem (Theorem 1.24), T is satisfied by an $\mathbf{S}$-structure M of cardinality $\leq \lambda$, and thus interpreting the constants by an enumeration $\vec{a}$ of M yields $(M, \vec{a}) \in \mathcal{W}$. $\qquad \square$

3.4. **Proof of the general omitting types theorem.** Note that if λ is an uncountable cardinal and $\Sigma(\bar{x})$ is λ-principal over T, where $\bar{x} = x_1, \ldots, x_n$ and $t_1(\bar{y}), \ldots, t_n(\bar{y})$ are S-terms, then the type $\Sigma(t_1(\bar{y}), \ldots, t_n(\bar{y}))$ is λ-principal over T. The following lemma shows that in basic continuous logic the converse of this fact holds.

Lemma 3.12. *If λ is an uncountable cardinal and $\Sigma(x_1, \ldots, x_n)$ is an $\mathbf{S}$-type of T such that $\Sigma(t_1(\bar{y}), \ldots, t_n(\bar{y}))$ is λ-principal over T, where $t_1(\bar{y}), \ldots, t_n(\bar{y})$ are S-terms, then $\Sigma(x_1, \ldots, x_n)$ is λ-principal over T.*

Proof. Let $\{\varphi_i(\bar{y})\}_{i \in I}$ generate $\Sigma(t_1(\bar{y}), \ldots, t_n(\bar{y}))$ over T. For each finite $J \subseteq I$, define $\psi_J(\bar{x})$ as

$$\exists \bar{y}\big(\bigwedge_{k \leq n} d(x_k, t_k(\bar{y})) \leq 0 \wedge \bigwedge_{i \in J} \varphi_i(\bar{y})\big).$$

We claim that the set of formulas of the form $\psi_J(\bar{x})$, where J is a finite subset of I, generates $\Sigma(\bar{x})$ over T. The proof is a standard compactness argument in basic continuous logic, but since we are not assuming previous experience with continuous logic, we include the details below.

Fix $\sigma(\bar{x}) \in \Sigma(\bar{x})$ and rationals r, r' such that $0 < r < r' < 1$. Since $T \cup \{\varphi_i(\bar{y})\}_{i \in I} \models_{\mathcal{L}_0, \mathbf{S}} \Sigma(t_1(\bar{y}), \ldots, t_n(\bar{y}))$, by compactness there exists a finite $J \subseteq I$ such that

$$(*) \qquad\qquad T \cup \big\{ \bigwedge_{i \in J} \varphi_i(\bar{y}) \big\} \models_{\mathcal{L}_0, \mathbf{S}} \sigma(t_1(\bar{y}), \ldots, t_n(\bar{y})) \geq r.$$

Since $\mathbf{S}$ includes uniform continuity moduli for all the predicate and operation symbols that occur in σ, there exists a rational $\delta \in (0, 1)$ such that

$$(**) \qquad \bigwedge_{k \leq n} d(x_k, t_k(\bar{y})) \leq \delta \wedge \sigma(t_1(\bar{y}), \ldots, t_n(\bar{y})) \geq r' \models_{\mathcal{L}_0, \mathbf{S}} \sigma(x_1, \ldots, x_n) \geq r.$$

By $(*)$ and $(**)$,

$$T \cup \big\{ \bigwedge_{k \leq n} d(x_k, t_k(\bar{y})) \leq \delta \wedge \bigwedge_{i \in J} \varphi_i(\bar{y}) \big\} \models_{\mathcal{L}_0, \mathbf{S}} \sigma(x_1, \ldots, x_n) \geq r.$$

Since σ is arbitrary and r is arbitrarily close to 1, this shows that the set of formulas of the form $\psi_J(\bar{x})$, where J is a finite subset of I, generates $\Sigma(\bar{x})$ over T. $\qquad \square$

We now prove some lemmas that connect principality with the logic topology $\tau_{\mathcal{L}}$ on $\mathrm{Str}_{\mathcal{L}}(\mathbf{S})$ (see Definition 2.1).

Recall that if X is a topological space and $x \in X$, then a λ-neighborhood of x is an intersection of less than λ neighborhoods of x. If A is a subclass of X, the λ-interior of A is the set of points in x that have a λ-neighborhood contained in A.

Lemma 3.13. *Let $\Sigma(\bar{x})$ be an* **S**-*type of a theory T with $\ell(\bar{x}) = n$, and let λ be an infinite cardinal.*

(1) *If λ is uncountable, then $\Sigma(\bar{x})$ is λ-principal over T if and only if the class $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}\big(T \cup \Sigma(\bar{x})\big)$ has nonempty λ-interior in $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T)$.*

(2) *If $\lambda = \omega$, then $\Sigma(\bar{x})$ is ω-principal over T if and only if there exist S-terms $t_1(\bar{y}), \ldots, t_n(\bar{y})$ such that the class $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}\big(T \cup \Sigma(t_1(\bar{y}), \ldots, t_n(\bar{y}))\big)$ has nonempty interior in $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(T)$.*

Proof. Assume that λ is uncountable and $\Sigma(\bar{x})$ is λ-principal over T, and let $\{\varphi_i(\bar{x})\}_{i<\mu}$ generate $\Sigma(\bar{x})$ over T, where $\mu < \lambda$. The class

$$\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(\varphi_i(\bar{x})) = \bigcap_{r\in\mathbb{Q}\cap(0,1)} \mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(\varphi_i(\bar{x}) > r)$$

is λ-open and hence so is

$$\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(\{\varphi_i(\bar{x})\}_{i<\mu}) = \bigcap_{i<\mu} \mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(\varphi_i(\bar{x})).$$

Since $T\cup\{\varphi_i(\bar{x})\}_{i<\mu}$ is satisfiable by an **S**-structure and $T\cup\{\varphi_i(\bar{x})\}_{i<\mu} \models_{\mathcal{L}_0} \mathbf{S}\ \Sigma(\bar{x})$, the class

$$\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T) \cap \mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}((\{\varphi_i(\bar{x})\}_{i<\mu})$$

is a nonempty λ-open subclass of $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T)$ contained in $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(\Sigma(\bar{x}))$.

Suppose, conversely, that $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T\cup\Sigma(\bar{x}))$ has nonempty λ-interior in $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T)$. Then there exist $\mu < \lambda$ and formulas $\varphi_i(\bar{x})$ for $i < \mu$ such that

$$\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T) \cap \bigcap_{i<\mu} \mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(\varphi_i(\bar{x}) > 0)$$

is a nonempty subclass of $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{x}\}}(T \cup \Sigma(\bar{x}))$. Choose $r_i \in \mathbb{Q}\cap(0,1)$ such that $T\cup\{\varphi_i(\bar{x})\geq r_i\}_{i<\mu}$ is satisfiable by an **S**-structure. Then,

$$T\cup\{\varphi_i(\bar{x})\geq r_i\}_{i<\mu} \models_{\mathcal{L}_0,\mathbf{S}} \Sigma(\bar{x}),$$

which finishes the proof of the uncountable case.

For the case $\lambda = \omega$, suppose that $T\cup\{\varphi(\bar{y})\}$ is satisfiable by an **S**-structure and $T\cup\{\varphi(\bar{y})\geq r\} \models_{\mathcal{L},\mathbf{S}} \Sigma(t_1(\bar{y}),\ldots,t_n(\bar{y}))$, where $t_1(\bar{y}),\ldots,t_n(\bar{y})$ are S-terms and $r\in\mathbb{Q}\cap(0,1)$. If $r'\in\mathbb{Q}\cap(r,1)$, the class $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(T)\cap\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(\varphi(\bar{y}) > r')$ is a nonempty open subclass of $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(T)$ contained in $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(\Sigma(t_1(\bar{y}),\ldots,t_n(\bar{y})))$.

To finish the proof for the countable case, assume that $\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(T\cup\Sigma(t_1(\bar{y}),\ldots,t_n(\bar{y})))$ has nonempty interior. Then there is a formula $\varphi(\bar{y})$ such that

$$\emptyset \neq \mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(T)\cap\mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(\varphi(\bar{y}) > 0) \subseteq \mathrm{Mod}_{\mathbf{S}\cup\{\bar{y}\}}(T\cup\Sigma(t_1(\bar{y}),\ldots,t_n(\bar{y}))).$$

As before, choose $r\in\mathbb{Q}\cap(0,1)$ such that $T\cup\{\varphi(\bar{y})\geq r\}$ is satisfiable by an **S**-structure. Then,

$$T\cup\{\varphi(\bar{y})\geq r\} \models_{\mathcal{L}_0,\mathbf{S}} T\cup\{\varphi(\bar{y}) > 0\} \models_{\mathcal{L}_0,\mathbf{S}} \Sigma(t_1(\bar{y}),\ldots,t_n(\bar{y})).$$

This shows that $\Sigma(\bar{x})$ is ω-principal over T. $\qquad\square$

Lemma 3.13 will be used to prove Theorem 3.6. However, we note the following important consequence of this lemma. Recall the definition of the logic $\mathcal{L}_1$ in Section 1.5.

Corollary 3.14. *Let T be theory in $\mathcal{L}_1$ and let λ be an infinite cardinal. An $\mathbf{S}$-type $\Sigma(\bar{x})$ is λ-principal over T with respect to all metric structures if and only if it is λ-principal over T with respect to complete metric structures. Moreover, Σ has a generator in $\mathcal{L}_0$ if and only if it has a generator in $\mathcal{L}_1$.*

Proof. The first assertion follows from Lemma 3.13 and Corollary 1.23. The "moreover" part is given by Lemma 3.13 and Proposition 2.6. $\qquad\square$

For the next lemma, we follow the notation used in Proposition 3.11. Let T be an S-theory. For $\mathbf{i} = i_1, \dots, i_n \in \lambda$, let $R_{T,\mathbf{i}}$ be the map

$$R_{T,\mathbf{i}} : \mathcal{W} \cap \mathrm{Mod}_{\mathbf{S} \cup C}(T) \to \mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}, \dots, c_{i_n}\}}(T)$$
$$(M, \vec{a}) \mapsto (M, a_{i_1}, \dots, a_{i_n}).$$

Lemma 3.15. *The map $R_{T,\mathbf{i}}$ is continuous, open and surjective.*

Proof. For notational simplicity, we will consider the case when $n = 1$. Continuity follows directly from the fact that any $(S \cup \{c_{i_1}\})$-sentence is also an $(S \cup C)$-sentence.

Now, with each $(S \cup C)$-sentence $\varphi = \varphi(c_1, \dots, c_{i_1}, \dots, c_m)$, with all the constants in C exhibited, let us associate the $(S \cup \{c_{i_1}\})$-sentence $\theta(c_{i_1})$ defined as

$$\forall x_1 \cdots \forall x_m \, \varphi(x_1, \dots, c_{i_1}, \dots, x_m).$$

To show that $R_{T,\mathbf{i}}$ is open and surjective, it suffices to show that $R_{T,\mathbf{i}}$ maps $\mathrm{Mod}_{\mathbf{S} \cup C}(\varphi)^c \cap \mathcal{W} \cap \mathrm{Mod}_{\mathbf{S} \cup C}(T)$ onto $\mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}\}}\big(\theta(c_{i_1})\big)^c \cap \mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}\}}(T)$.

Suppose $(M, \vec{a}) \in \mathrm{Mod}_{\mathbf{S} \cup C}(\varphi)^c \cap \mathcal{W} \cap \mathrm{Mod}_{\mathbf{S} \cup C}(T)$, and let $r \in \mathbb{Q} \cap (0, 1)$ be such that $(M, \vec{a}) \models_{\mathcal{L}_0} \varphi \leq r$. We certainly have $(M, a_{i_1}) \models_{\mathcal{L}_0} \theta(c_{i_1}) \leq r$, so $(M, a_{i_1}) \in \mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}\}}\big(\theta(c_{i_1})\big)^c \cap \mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}\}}(T)$.

Suppose, conversely, that $(M, a_{i_1}) \in \mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}\}}\big(\theta(c_{i_1})\big)^c \cap \mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}\}}(T)$, and let $r \in \mathbb{Q} \cap (0, 1)$ be such that $(M, a_{i_1}) \models_{\mathcal{L}_0} \theta(c_{i_1}) \leq r$. Pick $r' \in \mathbb{Q} \cap (r, 1)$. Then there are elements $a_k \in M$, for $k \leq m$ and $k \neq i_1$, such that $(M, a_1, \dots, a_m) \models_{\mathcal{L}_0} \varphi \leq r'$. Since $|T| \leq \lambda$, the downward Löwenheim-Skolem Theorem (Theorem 1.24) guarantees that there is an elementary substructure M_0 of M of cardinality $\leq \lambda$ containing $a_1, \dots, a_m$. Using the constants c_j with $j \notin \{i_1, \dots, i_n\}$ to name the remaining elements of M_0, we see that $(M, \vec{a}) \in \mathrm{Mod}_{\mathbf{S} \cup C}(\varphi)^c \cap \mathcal{W} \cap \mathrm{Mod}_{\mathbf{S} \cup C}(T)$. $\quad\square$

We now have the material we need to prove Theorem 3.6:

Proof of the omitting types theorem. Let T be theory of cardinality $\leq \lambda$ that is satisfied by an $\mathbf{S}$-structure and let $\{\Sigma_j(x_1, \dots, x_n)\}_{j < \lambda}$ be a set of types that are not λ-principal over T. By hypothesis in the countable case, and by Lemma 3.12 in the uncountable case, the types $\Sigma_j(t_1(\bar{y}), \dots, t_n(\bar{y}))$ are not principal over T, for any choice of S-terms $t_1(\bar{y}), \dots, t_n(\bar{y})$; hence, without loss of generality, we may assume that

(*) $\Sigma(t_1(\bar{y}), \dots, t_n(\bar{y}))$ is on the list whenever $\Sigma(x_1, \dots, x_n)$ is.

Let C be as before. By Lemma 3.13, for any $\mathbf{i} = i_1, \dots, i_n$ and any $j < \lambda$, the class

$$\mathrm{Mod}_{\mathbf{S} \cup \{c_{i_1}, \dots, c_{i_n}\}}\big(T \cup \Sigma_j(c_{i_1}, \dots, c_{i_n})\big)$$

is closed with empty λ-interior. Therefore, by Lemma 3.15, so is

$$\mathcal{C}_{T,\mathbf{i}} = R_{T,\mathbf{i}}^{-1}\big(\operatorname{Mod}_{\mathbf{S}\cup\{c_{i_1},\ldots,c_{i_n}\}}\big(T\cup\Sigma_j(c_{i_1},\ldots,c_{i_n})\big)\big).$$

Since $\mathcal{W}\cap\operatorname{Mod}_{S\cup C}(T)$ is nonempty and has the λ-Baire Property, there is

$$(M,\vec{a})\in\mathcal{W}\cap\operatorname{Mod}_{\mathbf{S}\cup C}(T)\setminus\bigcup_{\mathbf{i}}\mathcal{C}_{T,\mathbf{i}}.$$

Thus, for any $j < \lambda$, no subset of $\vec{a}$ realizes $\Sigma_j(\bar{x})$ in M. Furthermore, by our assumption $(*)$ above, no subset of $\langle\vec{a}\rangle$ realizes $\Sigma_j(\bar{x})$ in M. This means that $M\upharpoonright\langle\vec{a}\rangle$ omits each $\Sigma_j(\bar{x})$. The structure $M\upharpoonright\langle\vec{a}\rangle$ is a model of T because $M\upharpoonright\langle\vec{a}\rangle\preceq_{\mathcal{L}_0} M$, since $M\in\mathcal{W}$. $\qquad\square$

3.5. Omitting types in complete structures.

In Section 3.2 we observed that a type that is omitted by a structure may not be omitted by its metric completion. In this section we investigate the problem of omitting types in metrically complete structures. The background logic here is an extension of basic continuous logic, and $\mathbf{S}$ denotes a fixed signature with vocabulary S.

If $\Sigma(x_1,\ldots,x_n)$ is a type and $\delta\in\mathbb{Q}\cap[0,1]$, we denote by $\Sigma^\delta(x_1,\ldots,x_n)$ the type consisting of all the formulas of the form

$$\exists y_1\ldots\exists y_n\ \big(\ \bigwedge_{k\le n} d(x_k,y_k)\le\delta\wedge\sigma(y_1,\ldots,y_n)\ \big),$$

where σ ranges over all finite conjunctions of formulas in Σ.

Note that if $\bar{a} = a_1,\ldots,a_n$ realizes Σ in a structure M, then every point in the closed δ-ball of $\bar{a}$ realizes Σ^δ, that is, if $\bar{b} = b_1\ldots,b_n \in M$ is such that $\max_{k\le n} d(a_k,b_k)\le\delta$, then $\bar{b}$ realizes Σ^δ.

Definition 3.16. Let T be a consistent S-theory and let $\Sigma(\bar{x})$ be an $\mathbf{S}$-type of T.

(1) If λ is an uncountable cardinal, we will say that $\Sigma(\bar{x})$ is *metrically λ-principal* over T if Σ^δ is λ-principal over T for every $\delta > 0$.

(2) We will say that a Σ is *metrically ω-principal* over T if for every $\delta > 0$ there is a formula $\varphi(\bar{x})$ such that
    (a) $T\cup\{\varphi(\bar{x})\}$ is satisfiable by an $\mathbf{S}$-structure, and
    (b) $T\cup\{\varphi(\bar{x})\ge r\}\models_{\mathcal{L},\mathbf{S}}\Sigma^\delta(\bar{x})$, for some $r\in\mathbb{Q}\cap(0,1)$.

Proposition 3.17. *Let T be a consistent S-theory and let $\Sigma(\bar{x})$ be an $\mathbf{S}$-type of T. Then, for every infinite cardinal λ, the type Σ is metrically λ-principal over T if and only if Σ^δ is λ-principal over T for every $\delta > 0$.*

Proof. For uncountable λ the proposition is true by definition. For $\lambda = \omega$, we only need to prove that if $\Sigma(x_1,\ldots,x_n)$ is an $\mathbf{S}$-type of T such that $\Sigma(t_1(\bar{y}),\ldots,t_n(\bar{y}))$ is metrically ω-principal over T, where $t_1(\bar{y}),\ldots,t_n(\bar{y})$ are S-terms, then $\Sigma(x_1,\ldots,x_n)$ is metrically ω-principal over T. Using the uniform continuity moduli of $\mathbf{S}$ as in the proof of Lemma 3.12 one can see that for every $\epsilon\in\mathbb{Q}\cap(0,1)$ there exist $\delta,\rho\in\mathbb{Q}\cap(0,1)$ such that if $T\cup\{\varphi(\bar{x})\}$ is satisfiable by an $\mathbf{S}$-structure and

$$T\cup\{\varphi(\bar{y})\ge r\}\models_{\mathcal{L},\mathbf{S}}\Sigma^\delta(t_1(\bar{y}),\ldots,t_n(\bar{y})),$$

then

$$\exists\bar{u}\big(\ \bigwedge_{k\le n} d(x_k,t_k(\bar{u}))\le 0\wedge\varphi(\bar{u})\ \big)\ge\rho\models_{\mathcal{L},\mathbf{S}}\Sigma^\epsilon(x,\ldots,x).$$

$\qquad\square$

Proposition 3.18 (λ-omitting types property for complete structures)**.** *Let λ be any infinite cardinal. If T is an S-theory of cardinality $\leq \lambda$ that is satisfied by an* **S**-*structure and $\{\Sigma_j(\bar{x})\}_{j<\lambda}$ is a set of* **S**-*types that are not metrically λ-principal over T there is a model of T of cardinality $\leq \lambda$ whose metric completion omits each $\Sigma_j(\bar{x})$.*

Proof. If for each $j < \lambda$, the type $\Sigma_j(\bar{x})$ is not metrically principal over T, then by Proposition 3.17 for each $j < \lambda$ there exists $\delta_j > 0$ rational such that the types $\Sigma_j^{\delta_j}(\bar{x})$ is not principal over T. By the λ-omitting types property the types $\{\Sigma_j^{\delta_j}(\bar{x})\}_{j<\lambda}$ are simultaneously omitted by a structure M cardinality $\leq \lambda$. Notice now that if $\bar{a} \in \overline{M}$ realizes $\Sigma_j(\bar{x})$ and $\bar{a} = \lim_n \bar{a}_n$, where $\bar{a}_n \in M$ for each n, then $\bar{a}_n$ realizes $\Sigma_j^{\delta_j}(\bar{x})$ for infinitely many n, a contradiction. Thus $\overline{M}$ omits each $\Sigma_j(\bar{x})$. $\qquad\square$

Corollary 3.19 (λ-omitting types property for continuous logic)**.** *Let λ be any infinite cardinal. If T is an S-theory of continuous logic of cardinality $\leq \lambda$ that is satisfied by an* **S**-*structure and $\{\Sigma_j(\bar{x})\}_{j<\lambda}$ is a set of* **S**-*types of continuous that are not metrically λ-principal over T there is a complete model of T of cardinality $\leq \lambda$ that omits each $\Sigma_j(\bar{x})$.*

Proof. Immediate from Propositions 3.18 and 3.14. $\qquad\square$

If T is a complete S-theory, we can define a topology on the set of all complete **S**-types of T, as follows. If $p(\bar{x}), q(\bar{x})$ are such types, where $\bar{x} = x_1 \ldots, x_n$, we define $d(p,q)$ as the infimum of the set of real numbers r such that there exist a model M of T and tuples $\bar{a}, \bar{b} \in M$ satisfying $\max_{k\leq n} d(a_k, b_k) \leq r$. A compactness argument shows that d is a metric. Note that $d(p,q) \leq \delta$ if and only if $p^\delta \subseteq q$, where p^δ is defined as above. Hence, if $p(\bar{x})$ is metrically principal, $\delta > 0$, and M is an **S**-structure such that $M \models_{\mathcal{L}_0} T$, then there exists $q(\bar{x})$ such that $d(p,q) \leq \delta$ and q is realized in M. We use this to prove the following observation, due Henson:

Proposition 3.20. *If T is a complete S-theory and M is a complete* **S**-*structure such that $M \models_{\mathcal{L}_0} T$, then every complete* **S**-*type of T that is metrically principal is realized in M.*

Proof. Fix T and M as in the statement of the proposition, and let p be a complete **S**-type of T that is metrically principal. Using compactness and the preceding observations, we find, inductively, a sequence $(q_n)_{n<\omega}$ of complete **S**-types for T, a chain $M = M_0 \preceq_{\mathcal{L}_0} M_1 \preceq_{\mathcal{L}_0} M_2 \preceq_{\mathcal{L}_0} \ldots$ of models of T, and sequences $(\bar{a}_n)_{n<\omega}$, $(\bar{b}_n)_{n<\omega}$ such that for every $n < \omega$,

   (1) $q_n \supseteq p^{2^{-n}}$ (so $d(p, q_n) \leq 2^{-n}$),
   (2) $\bar{a}_n$ realizes q_n in M,
   (3) $\bar{b}_n$ realizes p in M_{n+1},
   (4) $d(a_n, b_n) \leq 2^{-n}$,
   (5) $d(a_{n+1}, b_n) \leq 2^{-(n+1)}$.

By (4) and (5) the sequences $(\bar{a}_n)_{n<\omega}$ and $(\bar{b}_n)_{n<\omega}$ are Cauchy and asymptotically equivalent (in $\bigcup_{n<\omega} M_n$). Since M is complete, their unique limit is in M, by (2). By (3), this limit realizes p. $\qquad\square$

Remark 3.21. By Propositions 3.18 and 3.20, if T is a complete S-theory and p is a complete **S**-type of T, then p is metrically principal if and only if p is realized

in every complete **S**-structure that is a model of T. Hence, the special case of Corollary 3.18 when $\lambda = \omega$ is Henson's omitting types theorem for complete metric structures [BYBHU08, Section 12], [BYU07, Section 1].

4. The Main Theorem

Recall that $\mathcal{L}_0$ denotes basic continuous logic (see Section 1.5) and $\mathcal{L}_1$ is its extension by all continuous connectives. We have proved (Theorem 3.6, Corollary 3.14) that $\mathcal{L}_0$ and $\mathcal{L}_1$ have the κ-omitting types property for every infinite cardinal κ. Łukasiewicz-Pavelka logic inherits trivially this property from $\mathcal{L}_1$, and continuous logic inherits it also for a stronger notion of non-principal type (Corollary 3.19). In this section we show that the κ-omitting types property characterizes $\mathcal{L}_0$. Analogous characterizations for Łukasiewicz-Pavelka logic and continuous logic follow as corollaries of the proof.

We prove that any $[0,1]$-valued logic $\mathcal{L}$ for continuous metric structures that extends $\mathcal{L}_0$ and has the κ-omitting types property for some uncountable regular cardinal κ is equivalent to $\mathcal{L}_0$, as long as only vocabularies of cardinality less than κ are considered. We will make some natural assumptions about $\mathcal{L}$, namely,

(1) Closure under the basic connectives, and under the existential quantifier (Definition 1.19),
(2) The finite occurrence property (Definition 1.9),
(3) Relativization to definable families of predicates (Definition 1.27),
(4) Every structure is equivalent in $\mathcal{L}$ to its metric completion (see Corollary 1.23).

The following is the main result of the paper:

Theorem 4.1 (Main Theorem). *Let $\mathcal{L}$ be a $[0,1]$-valued logic that satisfies properties* (1)–(4) *above and has the κ-omitting types property for some uncountable regular cardinal κ. If $\mathcal{L}$ extends $\mathcal{L}_0$, then $\mathcal{L}$ is equivalent to $\mathcal{L}_0$ for signatures of cardinality less than κ.*

Fix a cardinal κ as given by the statement of Theorem 4.1, let **S** be a signature of cardinality less than κ, and let us view **S**-sentences a $[0,1]$-valued functions on the class of **S**-structures, by identifying an **S**-sentence φ with the function $M \mapsto \varphi^M$. The equivalence stated by Theorem 4.1 means that the logic topologies $\tau_{\mathcal{L}}$ and $\tau_{\mathcal{L}_0}$ on the class of **S**-structures coincide, so, for every **S**-sentence φ of $\mathcal{L}$, the class $\mathrm{Mod}_{\mathbf{S}}(\varphi)$ is $\tau_{\mathcal{L}_0}$-closed. This implies that every **S**-sentence φ of $\mathcal{L}$ (viewed as a $[0,1]$ valued function) is $\tau_{\mathcal{L}_0}$-continuous, for if $[r,s]$ is a subinterval of $[0,1]$ with rational endpoints, then $\varphi^{-1}[r,s] = \mathrm{Mod}_{\mathbf{S}}(\varphi \geq r) \cap \mathrm{Mod}_{\mathbf{S}}(\varphi \leq s)$, which is then $\tau_{\mathcal{L}_0}$-closed. Since $\tau_{\mathcal{L}_0}$ is compact, by the Stone-Weierstrass Theorem for lattices, every sentence of $\mathcal{L}$ can be approximated, uniformly over the class of **S**-structures, by sentences of $\mathcal{L}_0$.

The rest of this section is devoted to the proof of the Main Theorem. The strategy of the proof is to show that if $\mathcal{L}$ extends $\mathcal{L}_0$ strictly, then there exist structures that are metrically isomorphic, but nonequivalent in $\mathcal{L}$; this contradicts the Isomorphism Property of $\mathcal{L}$ (see Definition 1.3).

The section is divided into two parts. In the first one we use the κ-omitting types property of the logic $\mathcal{L}$ to prove that $\mathcal{L}$ is λ-compact for every $\lambda < \kappa$ (Proposition 4.4), and the second part of the section is devoted to the proof of the Main Theorem.

4.1. **Obtaining compactness from the omitting types property.** Here, $\mathcal{L}$ denotes a $[0,1]$-valued logic that satisfies the hypotheses of the Main Theorem (Theorem 4.1) and κ denotes an uncountable regular cardinal such that $\mathcal{L}$ has the κ-omitting types property.

Recall that if M is an S-structure, an $[0,1]$-valued predicate $R^M(\bar{x})$ is discrete if R^M only takes on values in $\{0,1\}$. As we observed in Definition 1.26, if R is a predicate in S, then the interpretation R^M is discrete if and only if

$$M \models_{\mathcal{L}} \forall \bar{x} \; \text{Discrete}\big(R(\bar{x})\big),$$

where Discrete$\big(R(\bar{x})\big)$ is an abbreviation of the sentence $R(\bar{x}) \vee \neg R(\bar{x})$.

Definition 4.2. Let M be a structure, P a new monadic predicate symbol, and $\lhd$ a new binary predicate symbol. We will say that $(P^M, \lhd^M)$ is a *discrete linear ordering* if $M \models_{\mathcal{L}} \theta$, where θ is the conjunction of the following sentences:

- $\forall x \; \text{Discrete}\big(P(x)\big)$
- $\forall x,y \, \big(\neg P(x) \vee \neg P(y) \vee \text{Discrete}(x \lhd y)\big)$
- $\forall x,y \, \big(\neg P(x) \vee \text{Discrete}(d(x,y))\big)$
- $\forall x \, \big[\neg P(x) \vee \neg (x \lhd x)\big]$
- $\forall x,y \, \big[\neg\big(P(x) \wedge P(y) \wedge (x \lhd y)\big) \vee \neg(y \lhd x)\big]$
- $\forall x,y,z \, \big[\neg\big(P(x) \wedge P(y) \wedge (x \lhd y) \wedge (y \lhd z)\big) \vee (x \lhd z)\big]$
- $\forall x,y \, \big[\neg\big(P(x) \wedge P(y)\big) \vee \big((x \lhd y) \vee (y \lhd x) \vee \neg d(x,y)\big)\big].$

Notice that if M be a structure, P is monadic predicate symbol, and $(P^M, \lhd^M)$ is a discrete linear ordering, then $(P^M, \lhd^M)$ is a linear ordering in the usual sense.

Lemma 4.3. *Let T be a consistent $\mathcal{L}$-theory of cardinality $\leq \kappa$. If T has a model M such that $(P^M, \lhd^M)$ is a discrete linear ordering with no right endpoint, then T has a model N such that $(P^N, \lhd^N)$ is a discrete linear ordering of cofinality κ.*

Proof. Let $(c_i)_{i<\kappa}$ be a family of new constants. Let θ be the sentence given in Definition 4.2. Define

$$T' = T \cup \{\theta\} \cup \big\{\, \forall x \, (\neg P(x) \vee \exists y \, (P(y) \wedge (x \lhd y)\,)\,)\,\big\}$$

$$\cup \{P(c_i)\}_{i<\kappa} \cup \{(c_i \lhd c_j) \vee \neg d(c_i, c_j)\}_{i<j<\kappa}.$$

We first claim that T' is satisfiable. To see this, notice that if M is a model of T such that $(P^M, \lhd^M)$ is a discrete linear ordering with no right endpoint, $a \in P^M$, and $c_i^M = a$ for all $i < \kappa$, then $(M, c_i^M)_{i<\kappa} \models_{\mathcal{L}} T'$.

Next, we claim that the type

$$\Sigma(x) = \{P(x)\} \cup \{\, c_i \lhd x \vee \neg d(c_i, x)\, \}_{i<\kappa}$$

is not κ-principal over T'. Suppose the contrary, and assume that $\Phi(x)$ generates of $\Sigma(x)$ over T'. Take a structure M and an element a of M such that $M \models_{\mathcal{L}} T' \cup \Phi[a]$. If $M \models_{\mathcal{L}} \neg P[a]$, then $M \not\models_{\mathcal{L}} \Sigma[a]$, so $\Phi(x)$ cannot generate $\Sigma(x)$ over T'. If $M \models_{\mathcal{L}} P[a]$, since $|\Phi(x)| < \kappa$ and $\mathcal{L}$ has the finite occurrence property, there is $j < \kappa$ such that c_j does not occur in $\Phi(x)$. Since $M \models_{\mathcal{L}} \theta \wedge P[a]$, we can find an interpretation c_j^M of c_j in M such that $a \lhd c_j^M$. But then $M \not\models_{\mathcal{L}} \Sigma[a]$, so $\Phi(x)$ cannot generate $\Sigma(x)$ over T.

Thus $\Sigma(x)$ is not κ-principal over T'. Since $\mathcal{L}$ has the κ-omitting types property, there is a model N of T' that omits $\Sigma(x)$. This means that no element of P^N is an upper bound of $\{c_i^N\}_{i<\kappa}$, i.e., the sequence $(c_i^N)_{i<\kappa}$ is cofinal in P^N. The result then follows since κ is regular. $\square$

Proposition 4.4. *The logic $\mathcal{L}$ is λ-compact for every $\lambda < \kappa$.*

Proof. We prove the proposition by induction on all $\lambda < \kappa$. Fix $\lambda < \kappa$ and suppose $\mathcal{L}$ is μ-compact for every $\mu < \lambda$. Let $\mathbf{S} = (S, \mathcal{U})$ be a signature, and let $T = \{\varphi_i\}_{i<\lambda}$ be an $\mathcal{L}$-theory such that every finite subset of T is satisfied by an $\mathbf{S}$-structure. We wish to show that T is satisfied by an $\mathbf{S}$-structure.

Let $\mathbf{S_0}$ be the signature $(S_0, \mathcal{U}_0)$, where S_0 is the subvocabulary of S formed by the symbols of S that occur in T and $\mathcal{U}_0$ is the restriction of $\mathcal{U}$ to S_0. By the Reduct Property of logics (Definition 1.3), it suffices to show that T is satisfied by an $\mathbf{S_0}$-structure. By the assumption that $\mathcal{L}$ has the finite occurrence property, we have $|S_0| \leq \lambda$, so without loss of generality we can assume that the uniform continuity moduli specified by $\mathcal{U}_0$ are made explicit by sentences in T, and hence every S_0-structure that satisfies T is bound to be $\mathbf{S_0}$-structure.

Let S^+ be a vocabulary that results from adding to S a unary predicate symbol P, two new binary predicate symbols, R and $\vartriangleleft$, and a family $(c_i)_{i<\lambda}$ of new constant symbols.

At this point we invoke the assumption (given on page 25) that the logic $\mathcal{L}$ permits relativization to definable families of predicates: for each S-sentence φ and each $i < \lambda$, let $\varphi_i^{\{y \mid R(x,y)\}}(x)$ be a relativization of φ_i to $\{y \mid R(x,y)\}$ (see Definition 1.27). Define an S^+-theory T^+ by letting

$$T^+ = \{\theta\} \cup \{P(c_i)\}_{i<\lambda}$$
$$\cup \left\{ \forall x \left(\forall y \left(R(x,y) \vee \neg R(x,y) \right) \wedge \left(\neg(c_i \vartriangleleft x) \vee \varphi_i^{\{y \mid R(x,y)\}}(x) \right) \right) \right\}_{i<\lambda},$$

where θ is as in Definition 4.2. We first claim that T^+ has a model M such that $(P^M, \vartriangleleft^M)$ is a discrete linear ordering with no right endpoint. By the induction hypothesis, for each $j < \lambda$, the theory $\{\varphi_i\}_{i<j}$ has a model M_j. Let M be the S^+-structure defined as follows:

- We have $c_i^M \neq c_j^M$ if $i < j < \lambda$, and the universe of M is the disjoint union of $\bigsqcup_{i<\lambda} M_i$ and $\{c_i^M\}_{i<\lambda}$.
- The distance between elements of M in the same M_i is as given by the metric of M_i, and between distinct elements of M not in the same M_i it is 1.
- If Q is a n-ary predicate symbol of S, the interpretation Q^M is $\bigsqcup_{i<\lambda} T^{M_i}$ in $\bigsqcup_{i<\lambda} M_i^n$ and 0 in $M^n \setminus \bigsqcup_{i<\lambda} M_i^n$.
- If f is a n-ary operation symbol of S, and $\bar{a} \in M^n$, then $f^M(\bar{a})$ is $f^{M_i}(\bar{a})$ if $a \in M_i^n$ and c_0 if $\bar{a} \in M^n \setminus \bigsqcup_{i<\lambda} M_i^n$.
- P^M is the characteristic function of $\{c_i^M\}_{i<\lambda}$.
- $\vartriangleleft^M$ is the characteristic function of $\{ (c_i^M, c_j^M) \mid i < j \}$.
- R^M is the characteristic function of $\bigcup_{i<\lambda}\{c_i\} \times M_i$.

By the renaming property of $\mathcal{L}$ (see Definition 1.3), M is a model of T^+. Note that $(P^M, \vartriangleleft^M)$ is a discrete linear ordering with no right endpoint. Thus, by Lemma 4.3, there is a model N of T^+ such that $(P^N, \vartriangleleft^N)$ is a discrete linear ordering of cofinality κ. Since $\lambda < \kappa$, there is $a \in N$ such that $c_i^N \trianglelefteq a^N$ for every $i < \lambda$. Thus, $N \restriction \{ b \mid N \models_{\mathcal{L}} R[a,b] \}$ is a model of T. $\square$

4.2. **Proof of the Main Theorem.** Recall that $\mathcal{L}$ denotes a $[0,1]$-valued logic that satisfies the hypotheses of the Main Theorem (Theorem 4.1) and κ denotes an uncountable regular cardinal such that $\mathcal{L}$ has the κ-omitting types property.

Since, by Proposition 4.4, $\mathcal{L}$ is λ-compact for every $\lambda < \kappa$, Proposition 2.4 provides the following result:

Proposition 4.5. *Let* $\mathbf{S}$ *be a signature with* $|S| < \kappa$. *If* $\equiv_{\mathcal{L}_0} \Rightarrow \equiv_{\mathcal{L}}$ *for* $\mathbf{S}$-*structures, then* $\mathcal{L}$ *is equivalent to* $\mathcal{L}_0$ *for* $\mathbf{S}$-*structures.* □

Thus, all that remains in order to prove the Main Theorem is to show that $\equiv_{\mathcal{L}_0} \Rightarrow \equiv_{\mathcal{L}}$ for signatures of cardinality less than κ.

If S is a vocabulary and M_0, M_1 are S-structures, we form the combined structure $[M_0, M_1]$ in the following way. For each n-ary predicate symbol R of S let R^0, R^1 be two distinct n-ary predicate symbols and for each n-ary operation symbol f of S let f^0, f^1 be two distinct n-ary operation symbols. Let P_0, P_1 be new monadic predicates. For $k = 0, 1$ let

$$S^k = \{\, R^k \mid R \text{ in } S \,\} \cup \{\, f^k \mid f \text{ in } S \,\} \cup \{P_k\}.$$

Then $[M_0, M_1]$ is the $(\{S^0\} \cup \{S^1\})$-structure whose universe is the disjoint union of the universes M_0 and M_1 (with the distance between elements of M_0 and elements of M_1 being 1) and such that

  · P_k^M is the characteristic function of M_k for $k = 0, 1$.
  · For every n-ary predicate symbol R of S and every $\bar{a} \in M^n$,

$$(R^k)^{[M_0, M_1]}(\bar{a}) = \begin{cases} R^{M_k}(\bar{a}), & \text{if } \bar{a} \in M_k^n \\ 0, & \text{otherwise.} \end{cases}$$

  · For every n-ary operation symbol f of S and every $\bar{a} \in M^n$,

$$(f^i)^{[M_0, M_1]}(\bar{a}) = \begin{cases} f^{M_k}(\bar{a}), & \text{if } \bar{a} \in M_k^n \\ a, & \text{otherwise,} \end{cases}$$

where a is a fixed element of M.

Now, the assumption that the logic $\mathcal{L}$ permits relativization to discrete predicates (see page 25) allows us to fix for every S-sentence φ an S^k-sentence φ^k such that $[M_0, M_1] \models_{\mathcal{L}} \varphi^k$ if and only if $M_k \models_{\mathcal{L}} \varphi$.

Proof of the Main Theorem. As observed above, we only have to show $\equiv_{\mathcal{L}_0} \Rightarrow \equiv_{\mathcal{L}}$ for signatures of cardinality less than κ. Suppose that this is not the case, and fix a signature $\mathbf{S}$ of cardinality less than κ, and $\mathbf{S}$-structures M_0, M_1 such that $M_0 \equiv_{\mathcal{L}_0} M_1$ and

$$(\dagger) \qquad\qquad M_0 \models_{\mathcal{L}} \gamma \quad \text{but} \quad M_1 \models_{\mathcal{L}} \gamma \leq r$$

for some $\mathcal{L}$-sentence γ and some $r \in \mathbb{Q} \cap (0, 1)$. Our goal is to show that these structures can be taken to be metrically isomorphic; by ($\dagger$), this would contradict property (3) of Definition 1.10.

Since $\mathcal{L}$ has the finite occurrence property, we may assume that the vocabulary S is finite. Let $\{c_i\}_{i<\kappa}$ be a set of new constants and for $\mathbf{i} = i_1, \ldots, i_n \in \kappa$, denote $c_{i_1}, \ldots, c_{i_n}$ by $c_{\mathbf{i}}$. For each $X \subseteq \kappa$, let $S_X = S^0 \cup S^1 \cup \{c_i^0, c_i^1\}_{i \in X}$. Define an S_κ-theory T as follows:

$$T = \{\gamma^0\} \cup \{\gamma^1 \leq r\} \cup \{P_0(c_i^0)\}_{i<\kappa} \cup \{P_1(c_i^1)\}_{i<\kappa} \cup$$
$$\{\, \psi^0(c_{\mathbf{i}}^0) \rightarrow_{\mathrm{L}} (\psi^1(c_{\mathbf{i}}^1) \geq s) \mid \psi(\bar{x}) \text{ an } S\text{-formula of } \mathcal{L}_0,$$
$$\mathbf{i} \text{ in } \kappa \text{ with } \ell(\mathbf{i}) = \ell(\bar{x}), s \in \mathbb{Q} \cap (0, 1) \,\}.$$

Our initial goal is to show that T is consistent. In order to do so, it is sufficient to show that the S_1-theory

$$T_1 = \{\gamma^0\} \cup \{\gamma^1 \le r\} \cup \{P_0(c_0^0)\} \cup \{P_1(c_0^1)\}\cup$$
$$\big\{\, \psi^0(c_0^0) \to_L (\psi^1(c_0^1) \ge s) \mid \psi(x) \text{ an } S\text{-formula of } \mathcal{L}_0,\ s \in \mathbb{Q} \cap (0,1) \,\big\}.$$

is consistent, since any model N of T_1 can be expanded to a model of T by defining $(c_i^0)^N = (c_0^0)^N$ and $(c_i^1)^N = (c_0^1)^N$ for $i < \kappa$.

Now, T_1 is countable since S is finite, so by the ω-compactness of $\mathcal{L}$, we need only show that every finite subset of T_1 has a model.

Claim. Let $\{\psi_k(x)\}_{k \le m}$ be a finite set of S-formulas of $\mathcal{L}_0$ and let $s \in \mathbb{Q} \cap (0,1)$ be given. Then for every $a \in M_0$ there is $b \in M_1$ such that

$$[M_0, M_1] \models_{\mathcal{L}} P_0[a] \wedge P_1[b] \wedge \bigwedge_{k \le m} (\psi_k^0[a] \to_L (\psi_k^1[b] \ge s)).$$

Proof of the claim. Fix $a \in M_0$ and $s \in \mathbb{Q} \cap (0,1)$. For each $k \le m$ choose $r_k, t_k \in \mathbb{Q} \cap (0,1)$ such that

$$(\psi_k[a])^{M_0} - (1 - s) \le r_k < t_k \le (\psi_k[a])^{M_0}$$

and set $\epsilon = \min_{k \le m} t_k - r_k$. Since $M_0 \models_{\mathcal{L}} \exists x (\bigwedge_{k \le m} (\psi_k(x) \ge t_k))$, the same sentence holds in M_1, thus there is b in M_1 such that $(\bigwedge_{k \le m} (\psi_k[b] \ge t_k))^{M_1} \ge 1 - \epsilon$. For each k,

$$(\psi_k[b] \ge t_k)^{M_1} \ge 1 - \epsilon \ge 1 - (t_k - r_k).$$

Hence, by Proposition 1.16,

$$(\psi_k[b])^{M_1} \ge t_k + 1 - (t_k - r_k) - 1 = r_k \ge (\psi_k[a])^{M_0} - (1 - s) = s + (\psi_k[a])^{M_0} - 1.$$

By the same proposition, this yields $(\psi_k[b] \ge s)^{M_1} \ge (\psi_k[a])^{M_0}$. Thus,

$$[M_0, M_1] \models_{\mathcal{L}} \bigwedge_{k \le m} (\psi_k^0[a] \to_L (\psi_k[b] \ge s)).$$

By the claim, every finite subset of T_1 is satisfied by an expansion by constants of $[M_0, M_1]$; hence, by the ω-compactness of $\mathcal{L}$, the theory T_1 has a model. As observed above, every such model yields a model of T.

Our next (and final) goal is to show that T has a model $[\hat{M}_0, \hat{M}_1]$ such that the set $\{(c_i^0)^{\hat{M}_0}\}_{i < \kappa}$ is dense in $\hat{M}_0$, and the set $\{(c_i^1)^{\hat{M}_1}\}_{i < \kappa}$ is dense in $\hat{M}_1$. Once this is accomplished, the definition of T ensures that the map

$$(c_i^0)^{\hat{M}_0} \mapsto (c_i^1)^{\hat{M}_1}$$

is a metric isomorphism between a dense subset of $\hat{M}_0$ and a dense subset of $\hat{M}_1$. Since all the predicates in S are uniformly continuous with respect to the distinguished metric, our isomorphism can be uniquely extended to a metric isomorphism between the completion of $\hat{M}_0$ and the completion of $\hat{M}_1$. But since, by assumption, every structure is equivalent in $\mathcal{L}$ to its completion, the completion of $\hat{M}_0$ satisfies γ, whereas the completion of $\hat{M}_1$ satisfies $\gamma \le r$, which contradicts the isomorphism property of $\mathcal{L}$ (see Definition 1.3).

By the preceding remark, all that remains to show is that there is a model of T that omits all the types

$$\Sigma_\epsilon(x) = \big\{\, (P_0(x) \wedge d(x, c_i^0) \ge \epsilon) \vee (P_1(x) \wedge d(x, c_j^1) \ge \epsilon) \,\big\}_{i,j < \kappa} \qquad (\epsilon \in \mathbb{Q} \cap (0,1)).$$

Since $\mathcal{L}$ has the κ-omitting types property, it suffices to show that, for each $\epsilon \in \mathbb{Q} \cap (0,1)$, the type $\Sigma_\epsilon(x)$ is not κ-principal over T. Suppose that $\Sigma_\epsilon(x)$ is κ-principal for some ϵ, and let $\Phi(x)$ generate $\Sigma_\epsilon(x)$ over T.

Fix $\delta \in \mathbb{Q} \cap (0,\epsilon)$. For $X \subseteq \kappa$ let us denote $T \upharpoonright S_X$ as T_X, and let

$$X_0 = \{0\} \cup \{\, j \mid c_j^0 \text{ or } c_j^1 \text{ occurs in } \Phi(x) \,\}.$$

Since $|\Phi(x)| < \kappa$ and $\mathcal{L}$ has the finite occurrence property, there is $j_1 \in \kappa \setminus X_0$. Let $X_1 = X_0 \cup \{j_1\}$. We now use the $|\Phi|$-compactness of $\mathcal{L}$ to show that the set

$$(\ddagger) \qquad\qquad T_{X_1} \cup \Phi(x) \cup \{\neg P_0(x) \vee d(x, c_{j_1}^0) \leq \delta\}$$

is satisfiable. Since $\Phi(x)$ generates $\Sigma_\epsilon(x)$, there is an S_κ-structure $[M_0', M_1']$ and an element a of $[M_0', M_1']$ such that $[M_0', M_1'] \models_{\mathcal{L}} T \cup \Phi[a]$. If $a \in M_1'$, the satisfiablity of $(\ddagger)$ is immediate (since $\neg P_0^{[M_0', M_1']}(a) = 1$), so suppose $a \in M_0'$. Since $M_0 \equiv_{\mathcal{L}_0} M_1$, the argument used to prove our claim above shows that

$(\diamond)$ for every finite set $\{\psi_k(\bar{x}, y)\}_{k \leq m}$ of S-formulas of $\mathcal{L}_0$ and every $s \in \mathbb{Q} \cap (0,1)$ there is b in M_1' such that whenever $\ell(\mathbf{i})$ is a list of indices in X_1 with $\ell(\mathbf{i}) = \ell(\bar{x})$, the structure $[M_0', M_1']$ satisfies

$$\bigwedge_{k \leq m} (\psi_k^0(c_{\mathbf{i}}^0, y)[a] \to_{\mathrm{L}} (\psi_k^1(c_{\mathbf{i}}^1, y)[b] \geq s)).$$

Let $\Gamma(c_{\mathbf{i}}^0, c_{\mathbf{i}}^1, c_{j_1}^0, c_{j_1}^1)$ be a finite subset of T_{X_1}, where $\mathbf{i} \in X_0$ and all the new constants are being displayed, and let $S(\Gamma)$ denote the finite part of S_{X_0} that occurs in Γ. Notice that the reduct $[M_0', M_1'] \upharpoonright S(\Gamma)$ satisfies $\Gamma \upharpoonright S_{X_0}$. Let

$$\psi_0(\bar{x}, y), \ldots, \psi_m(\bar{x}, y)$$

be a list of all the S-formulas such that the implications

$$\psi_0^0(c_{\mathbf{i}}^0, c_{j_1}^0) \to_{\mathrm{L}} (\psi_0^1(c_{\mathbf{i}}^1, c_{j_1}^1) \geq s_0), \ldots, \psi_m^0(c_{\mathbf{i}}^0, c_{j_1}^0) \to_{\mathrm{L}} (\psi_m^1(c_{\mathbf{i}}^1, c_{j_1}^1) \geq s_m)$$

occur in Γ, let $s = \min_{k \leq m} s_k$, and fix $b \in M_1'$ corresponding to $\{\psi_k(\bar{x}, y)\}_{k \leq m}$ and s as given by $(\diamond)$. Now let

$$([M_0', M_1'] \upharpoonright S(\Gamma), a, b)$$

denote the expansion of $[M_0', M_1'] \upharpoonright S(\Gamma)$ to S_{X_1} where a is the interpretation of $c_{j_1}^0$ and b is interpretation of $c_{j_1}^1$. Then, by $(\diamond)$, we have

$$\big([M_0', M_1'] \upharpoonright S(\Gamma), a, b\big) \models_{\mathcal{L}} \Gamma(c_{\mathbf{i}}^0, c_{\mathbf{i}}^1, c_{j_1}^0, c_{j_1}^1).$$

By the choice of a, $[M_0', M_1'] \models_{\mathcal{L}} T \cup \Phi[a]$, and trivially, we also have

$$\big([M_0', M_1'] \upharpoonright S(\Gamma), a, b\big) \models_{\mathcal{L}} d(x, c_{j_1}^0)[a] \leq \delta.$$

Therefore a realizes

$$\Gamma(c_{\mathbf{i}}^0, c_{\mathbf{i}}^1, c_{j_1}^0, c_{j_1}^1) \cup \Phi(x) \cup \{\neg P_0(x) \vee d(x, c_{j_1}^0) \leq \delta\}$$

in the structure $\big([M_0', M_1'] \upharpoonright S(\Gamma), a, b\big)$. Since $\mathcal{L}$ is $|\Phi|$-compact and Γ is an arbitrary finite subset of T_{X_1}, this shows that $(\ddagger)$ is satisfiable.

Fix now $j_2 \in \kappa \setminus X_1$, and let $X_2 = X_1 \cup \{j_2\}$. An argument symmetric to that which produced a model of $(\ddagger)$ shows that the theory

$$T_{X_2} \cup \Phi(x) \cup \{\, (\neg P_0(x) \vee d(x, c_{j_1}^0) \leq \delta) \wedge (\neg P_1(x) \vee d(x, c_{j_1}^1) \leq \delta) \,\}$$

is satisfied by an S_{X_2}-structure. To conclude the proof, we only need to expand this model to an S_κ-structure, i.e., we need to find interpretations for the constants c_i^0, c_i^1 with $i \in \kappa \setminus X_2$ in such a way that

$$T \cup \Phi(x) \cup \{ (\neg P_0(x) \vee d(x, c_{j_1}^0) \leq \delta) \wedge (\neg P_1(x) \vee d(x, c_{j_1}^1) \leq \delta) \}$$

is still satisfied; but this can be done by simply giving c_i^0, c_i^1, for $i \in \kappa \setminus X_2$, the same interpretation as c_0^0, c_0^1. Since $\delta < \epsilon$,

$$T \cup \Phi(x) \not\models_{\mathcal{L}} \Sigma_\epsilon(x),$$

so $\Phi(x)$ does not generate $\Sigma_\epsilon(x)$, as presumed. This concludes the proof that $\Sigma_\epsilon(x)$ is not κ-principal over T, and thus the proof of the Main Theorem.   □

The preceding proof is a refinement of the proof of the Main Theorem in [Lin78].

Remark 4.6. The κ-omitting types property for a theory T states that for every set of at most κ types that are not κ-principal over T there is a model of T that omits all the types in the set. In the proof of Theorem 4.1, we needed only a weak version of this property, namely, we need the existence of a model of T that omits countable sets of types that are not κ-principal. Thus, in basic continuous logic, the κ-omitting types property is equivalent to this apparently weaker version of it.

We conclude by observing that the argument used to prove Theorem 4.1 yields an analogous characterization of Łukasiewicz-Pavelka and of logic continuous logic:

Corollary 4.7. *Let $\mathcal{L}$ be a $[0,1]$-valued logic.*
  (1) *If $\mathcal{L}$ satisfies properties (1)–(4) of page 25, $\mathcal{L}$ extends Łukasziewicz-Pavelka logic, and there exists an uncountable regular cardinal κ such that $\mathcal{L}$ satisfies the κ-omitting types property, then $\mathcal{L}$ is equivalent to Łukasziewicz-Pavelka logic for signatures of cardinality less than κ.*
  (2) *If $\mathcal{L}$ satisfies properties (1)–(3) of page 25, $\mathcal{L}$ extends continuous logic, and there exists an uncountable regular cardinal κ such that $\mathcal{L}$ satisfies the κ-omitting types property for complete structures (Proposition 3.18), then $\mathcal{L}$ is equivalent to continuous logic for signatures of cardinality less than κ.*

Proof. We note first that the methods used in the proof of Theorem 4.1 to produce new structures from old ones (i.e., the construction of M from $\{M_j\}_{j<\lambda}$ in the proof of Proposition 4.4 and the construction of $[M_0, M_1]$ from M_0 and M_1 on page 28) yield complete structures from complete structures and 1-Lipschitz structures from 1-Lipschitz structures. Hence, (1) follows by assuming throughout the proof of Theorem 4.1 that all the structures involved are 1-Lipschitz. Moreover, it may be verified that the types omitted in the course of the proof are not metrically κ-principal; hence (2) follows by assuming that all the structures involved in the proof are complete and utilizing the metric κ-omitting types theorem for continuous logic.   □

REFERENCES

[BF85]   J. Barwise and S. Feferman (eds.), *Model-Theoretic Logics*, Springer-Verlag, New York, 1985. MR 87g:03033

[BYBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427. MR MR2436146

[BYU07]   Itaï Ben Yaacov and Alexander Usvyatsov, *On d-finiteness in continuous structures*, Fund. Math. **194** (2007), no. 1, 67–88. MR 2291717 (2007m:03080)

[BYU10]   _____, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5213–5259. MR 2657678

[Cai1]   Xavier Caicedo, *Lindström's theorem for positive logics, a topological view*, in *Logic Without Borders*, Roman Kossak, Juha Kontinen, Åsa Hirvonen, Andrés Villaveces, ed., Ontos Verlag, to appear.

[Cai2]   _____, *A Lindström's theorem for Lukasiewicz and continuous logic*, Manuscript.

[Cai93]   _____, *Compactness and normality in abstract logics*, Ann. Pure Appl. Logic **59** (1993), no. 1, 33–43. MR 1197204 (93m:03062)

[Cai95]   _____, *Continuous operations on spaces of structures*, Quantifiers: Logics, Models and Computation I, Synthese Library, vol. 248, 1995, pp. 263–296.

[Cai99]   _____, *The abstract compactness theorem revisited*, Logic and Foundations of Mathematics, Synthese Library, vol. 280, 1999, pp. 131–141.

[CK66]   Chen-Chung Chang and H. Jerome Keisler, *Continuous Model Theory*, Annals of Mathematics Studies, No. 58, Princeton Univ. Press, Princeton, N.J., 1966. MR MR0231708 (38 #36)

[GJ76]   Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Springer-Verlag, New York, 1976, Reprint of the 1960 edition, Graduate Texts in Mathematics, No. 43. MR 0407579 (53 #11352)

[GM04]   Marta García-Matos, *A framework for maximality and interpolation in abstract logics with and without negation*, Aspects of universal logic, Travaux Log., vol. 17, Univ. Neuchâtel, Neuchâtel, 2004, pp. 66–86. MR 2168187

[GMV05]   Marta García-Matos and Jouko Väänänen, *Abstract model theory as a framework for universal logic*, Logica universalis, Birkhäuser, Basel, 2005, pp. 19–33. MR MR2134728 (2005k:03091)

[Háj97]   Petr Hájek, *Fuzzy logic and arithmetical hierarchy. II*, Studia Logica **58** (1997), no. 1, 129–141. MR 1432158 (98j:03038)

[Háj98]   _____, *Metamathematics of fuzzy logic*, Trends in Logic—Studia Logica Library, vol. 4, Kluwer Academic Publishers, Dordrecht, 1998. MR 1900263 (2003c:03048)

[HI02]   C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110. MR 1 967 834

[HPS00]   Petr Hájek, Jeff Paris, and John Shepherdson, *Rational Pavelka predicate logic is a conservative extension of Lukasiewicz predicate logic*, J. Symbolic Logic **65** (2000), no. 2, 669–682. MR 1771076 (2001m:03048)

[Iov01]   José Iovino, *On the maximality of logics with approximations.*, J. Symbolic Logic **66** (2001), no. 4, 1909–1918.

[KM01]   H. Jerome Keisler and Arnold W. Miller, *Categoricity without equality*, Fund. Math. **170** (2001), no. 1-2, 87–106, Dedicated to the memory of Jerzy Łoś. MR 1881370 (2002k:03054)

[Lin69]   P. Lindström, *On extensions of elementary logic*, Theoria **35** (1969), 1–11. MR 39 #5330

[Lin78]   Per Lindström, *Omitting uncountable types and extensions of elementary logic*, Theoria **44** (1978), no. 3, 152–156. MR 586927 (82d:03062)

[MN06]   Petra Murinová and Vilém Novák, *Omitting types in fuzzy logic with evaluated syntax*, MLQ Math. Log. Q. **52** (2006), no. 3, 259–268. MR 2239466 (2007f:03032)

[Nov89]   Vilém Novák, *Fuzzy sets and their applications*, Adam Hilger Ltd., Bristol, 1989, Translated from the Czech. MR 1019090 (91a:04007)

[Nov90]   _____, *On the syntactico-semantical completeness of first-order fuzzy logic. II. Main results*, Kybernetika (Prague) **26** (1990), no. 2, 134–154. MR 1059796 (91f:03047b)

[Nov95]   _____, *A new proof of completeness of fuzzy logic and some conclusions for approximate reasoning*, Proc. Int. Conference FUZZ-IEEE/FES'95 (Yokohama), 1995.

[Pav79a] Jan Pavelka, *On fuzzy logic. I*, Z. Math. Logik Grundlag. Math. **25** (1979), no. 1, 45–52, Many-valued rules of inference. MR 524558 (80j:03038a)

[Pav79b] _____, *On fuzzy logic. II. Enriched residuated lattices and semantics of propositional calculi*, Z. Math. Logik Grundlag. Math. **25** (1979), no. 2, 119–134. MR 527904 (80j:03038b)

[Pav79c] _____, *On fuzzy logic. III. Semantical completeness of some many-valued propositional calculi*, Z. Math. Logik Grundlag. Math. **25** (1979), no. 5, 447–464. MR 543638 (80j:03038c)

Department of Mathematics, Universidad de los Andes, Apartado Aereo 4976, Bogotá, Colombia
E-mail address: xcaicedo@uniandes.edu.co

Department of Mathematical Sciences, Carnegie Mellon University, 5000 Forbes Ave, Pittsburgh, PA 15213, USA

Department of Mathematics, University of Toronto, Toronto, Ontario, CANADA M5S 2E4

Department of Mathematics, The University of Texas at San Antonio, One UTSA Circle, San Antonio, TX 78249-0664, USA
E-mail address: iovino@math.utsa.edu