Applications of Model Theory to Functional Analysis

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Preface to the Dover Edition

These notes evolved from a series of lectures given at Carnegie Mellon University during the late 1990's. The notes were later adapted to be used as textbook for a minicourse on applications of ultraproducts in analysis, given at the Universidad de los Andes campus in Mérida, Venezuela, in 2002. The audience in Pittsburgh was composed of logicians and analysts; the course in Mérida was taken by students from graduate programs in Latin America. The goal of the lectures was to show how basic ideas that evolved independently in two different fields of mathematics — in this case, model theory and Banach space theory — melded to yield beautiful results; the showcases here are two theorems of Banach space theory, both of which bear the name of Jean-Louis Krivine, namely, Krivine's Theorem on the finite representability of ℓ_p in all Banach lattices, and the Krivine-Maurey result that every stable Banach space contains some ℓ_p. At the time of preparing these notes there was no other elementary exposition in the literature that showed how the proofs of these theorems are related to fundamental ideas from logic, nor how the two proofs are related to each other. The same is true today, despite the explosion of activity in the areas of interaction between logic and functional analysis.

The original text has not been altered; only the historical remarks at the end have been edited slightly in order to bring them up to date. In the original edition, for the logical language we used Henson's formalism of approximations of formulas; today, the preference is to use real-valued logic (see the historical remarks). Nevertheless, as pointed out in Section 1.5, for the restricted class of model-theoretic types used here (that is, quantifier-free types), the equivalence between both approaches is immediate.

The first edition was dedicated to my wife Martha and our daughter Abigail, who at the time was a baby. Since then, our son Luca was born. The monograph is now dedicated to him as well.

To Martha, Abigail, and Luca

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Applications of Model Theory to Functional Analysis

CHAPTER 0

Introduction

If one were to compose a list of the most important results in of the last thirty years in Banach space theory, the following would have to be included:

1. Tsirelson's example of a Banach space not containing ℓ_p or c_0 [**Tsi74**],
2. Krivine's Theorem [**Kri76**],
3. The Krivine-Maurey theorem that every stable space contains some ℓ_p almost isometrically [**KM81**],
4. The Bourgain-Rosenthal-Schechtman proof that there are uncountably many complemented subspaces of L_p [**BRS81**],
5. Gowers' dichotomy [**Gow96, Gow02, Gow03**].

Apart from their importance, these results have in common the fact that they were proved by using concepts and techniques that originated in mathematical logic. Tsirelson's construction was inspired by set-theoretic forcing. Krivine's theorem was proved using classical model-theoretic tools such as types and indiscernible sequences. The Krivine-Maurey theorem was based on the notion of model-theoretic stability. The main tool of the Bourgain-Rosenthal-Schechtman paper is an ordinal-valued rank function of the type commonly used in model theory. Gowers' dichotomy was proved using Gowers' celebrated Ramsey theorem [**Gow96, Gow02, Gow03**], which resulted as a refinement of the methods used by Galvin-Prikry [**GP73**] and Ellentuck [**Ell74**] in proving partition theorems that emerged from problems about the existence of models of set theory with particular properties. (For a detailed historical account of this, see [**Lar12**].)

Rosenthal's ℓ_1 theorem [**Ros74**] is regarded as one of the most elegant theorems of Banach space theory. It was observed by Farahat [**Far74**] that Rosenthal's paper contains an independent proof of the classical theorem (proved by Nash-Williams in the 1960's [**NW68**], but independently also by Cohen and Ehrenfeucht, among others) that every open subset of $\mathbb{N}^{\mathbb{N}}$ (endowed with the product topology) is Ramsey. This observation unveiled infinite Ramsey theory as an important tool in Banach space theory and triggered a host of applications that peaked with Gowers' Ramsey theorem [**Gow96, Gow02, Gow03**].

For a detailed exposition of how combinatorial methods from set theory have influenced Banach space theory, we refer the reader to Todorčević's book on Ramsey spaces [**Tod10**].

In these notes, we will focus on a particular set of concepts where Banach space theory has made contact with logic, namely, concepts that originated in model theory. Among these are:

1. Ultraproducts,
2. Indiscernible sequences (called 1-*subsymmetric* sequences in Banach space theory),
3. Ordinal ranks (called *ordinal indices* in analysis),
4. Ehrenfeucht-Mostowski models (called *spreading models* in Banach space theory),
5. Spaces of types,
6. Stability.

Some of these concepts were introduced in analysis by direct adaptation of constructions from model theory (*e.g.*, Banach space ultrapowers and indiscernible sequences, introduced in Krivine's thesis [**Kri67**] and in the proof of Krivine's Theorem [**Kri76**], respectively); others were inspired by analogies (*e.g.*, Banach space stability, introduced by Krivine and Maurey in [**KM81**], motivated by the fact that in a stable theory every indiscernible sequence is totally indiscernible); and yet others were discovered independently by analysts (*e.g.*, spreading models — and their construction using Ramsey's Theorem — which were introduced by Brunel and Sucheston in the study of ergodic properties of Banach spaces; see [**BS74**]).

As we will see as well, certain basic notions from Banach space theory can be seen quite naturally from a model-theoretic perspective. An example is that of finite representability: a Banach space X is finitely representable in a Banach space Y (a Banach space-theoretic concept) if and only if Y is a model of the existential theory of X (a model-theoretic concept).

There are even similarities between diving lines in both fields; for example, there is an equivalence, in an abstract categorical sense, between the the reflexive/nonreflexive dichotomy in the class of all Banach spaces and the stable/unstable dichotomy in the class of all model-theoretic structures. (See [**Iov99c**].)

Therefore, it would be desirable to have bridges between these two fields so that techniques from one of them can become useful in the other. Some considerations must be taken into account, however:

(1) First-order logic, the traditional language of classical model theory, is not the natural logic to analyze Banach spaces as models. Banach space theory is carried out in higher order logics, as is functional analysis in general. Furthermore, the first-order theory of Banach spaces is known to be equivalent to a second order logic. (See [**SS78**].)

(2) The aforementioned concepts from Banach space theory are not literal translations of their first-order counterparts. For instance, a Banach space ultrapower of a Banach space X is not an ultrapower of X in the sense traditionally considered in model theory, and is not an elementary extension of X in the sense of first-order logic. However, there is a strong analogy between the role played by Banach space ultrapowers in Banach space theory and that played by algebraic ultrapowers in model theory.

Similarly, what is regarded in Banach space theory as the *space of types* of a space is not what is understood as the space of types in traditional first-order model theory. Let us recall the definition of type given by Krivine and Maurey [**KM81**]:

> Let X be a fixed separable Banach space. A *type* on X is a function $\tau \colon X \to \mathbb{R}$ such that there exists a sequence (x_n) in X satisfying
>
> $$\tau(x) = \lim_{n\to\infty} \|x + x_n\|, \qquad \text{for all } x \in X.$$

The space of types of X, as defined by Krivine and Maurey [**KM81**], is the set of types on X with the topology of pointwise convergence. This notion of space of types is motivated by the corresponding notion from first-order logic. A priori, the analogy is not entirely clear. However, as we shall see, both notions of type are intimately connected.

A formal framework for a model-theoretic analysis of Banach spaces was first introduced by Henson in [**Hen76**]. The scope of Henson's framework was expanded by Henson and the author [**HI02**], and later elegantly reformulated by Ben-Yaacov and Usvyatsov [**BYU10**] using the notion of continuous model theory developed by Chang and Keisler in the 1960's [**CK66**]. (See also [**BYBHU08**].)

The special feature of Henson's model-theoretic framework is that, although it is appropriate for structures from functional analysis, it preserves many of the desirable characteristics of first-order model theory, *e.g.*, the compactness theorem, Löwenheim-Skolem theorems, and omitting types theorem. (In fact, it provides a natural setting for the classification theory, in the sense of [**She90**], for structures from infinite dimensional analysis.) Furthermore, it provides a precise language for the translations and analogies mentioned previously. For example, Krivine-Maurey types correspond exactly to quantifier-free types in Henson's formalism.

The question of how the classical sequence spaces ℓ_p $(1 \leq p < \infty)$ and c_0 occur inside every Banach space has played a central role in Banach space geometry for more than half a century. The first example of a Banach space not containing ℓ_p or c_0 was constructed by Tsirelson [**Tsi74**]. Shortly after Tsirelson's example appeared in print, Krivine [**Kri76**] published the celebrated result, known now as Krivine's Theorem, that states that for every Banach space X there exists p with $1 \leq p \leq \infty$ such that ℓ_p is block-finitely representable in X.

The question then arises of what conditions on the norm of a Banach space guarantee that the space contains ℓ_p or c_0 almost isometrically. This problem is still open, but the most elegant partial answer known thus far is the theorem proved by Krivine and Maurey in [**KM81**] that states that every stable Banach space contains some ℓ_p almost isometrically.

In these notes we use the model-theoretic framework introduced by Henson to prove a general principle about block representability of ℓ_p in indiscernible sequences of the type that logicians call "Morley sequences". (Theorem 11.1). Both Krivine's Theorem and the Krivine-Maurey theorem about ℓ_p subspaces of stable spaces follow as consequences of this principle. In the original proofs (in [**Kri76**] and [**KM81**]), the model-theoretic connections are in the background; here, we attempt to bring them to the fore.

A note on the exposition. These notes are of introductory nature, since they were prepared with students in mind. We give pointers to the literature at the end, in the historical remarks.

The prerequisites in Banach space theory are minimal. We assume that the reader is familiar with the definition of the classical sequence spaces (ℓ_p and c_0) and with the definition of Banach space operator. For the prerequisites in logic, a beginning course in model theory (for example, the first three chapters of [**CK90**] or the first five chapters of [**Hod**]) will more than suffice.

The exercises are an integral part the text, and the reader is expected to work on all of them. Some exercises are indicated explicitly under the heading *Exercise*; however, most of them occur only tacitly, either as proofs for which only rough sketch is given (labeled *Sketch of proof*) and where the reader is expected to supply the missing details, or as remarks (under the heading *Remark*) in which no proof is given; in this case the reader should provide the entire supporting argument.

The historical remarks given at the end should be regarded as an integral part of the exposition.

A word about notation. Model theorists use the letters p, q, etc. to denote types. However, in Banach space theory, these letters are reserved to denote special parameters, namely, the parameter p in the $L_p(\mu)$ spaces. Since the two notational traditions clash here, we have used the letters t, t', etc. to denote types. For similar reasons, we have avoided using the letter T to denote theories, since in linear analysis it is customarily used to denote operators.

For a more comprehensive introduction to the model theory of linear structures, the reader is referred to [**HI02**].

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CHAPTER 1

Preliminaries: Banach Space Models

1. Banach Space Structures and Banach Space Ultrapowers

A Banach space is finite dimensional if and only if the unit ball is compact, *i.e.*, if and only if for every bounded family $(x_i)_{i\in I}$ and every ultrafilter $\mathcal{U}$ on the set I, the $\mathcal{U}$-limit

$$\lim_{i,\mathcal{U}} x_i$$

exists. If X is an infinite dimensional Banach space and $\mathcal{U}$ is an ultrafilter on a set I, there is a canonical way of expanding X to a larger Banach space $\hat{X}$ by adding for every bounded family $(x_i)_{i\in I}$ in X an element $\hat{x} \in \hat{X}$ such that $\|\hat{x}\| = \lim_{i,\mathcal{U}} \|x_i\|$. This is accomplished through the construction of *Banach space ultrapower*, which we now define.

Let $(X_i)_{i\in I}$ be a family of normed spaces. Define

$$\ell_\infty\big(\prod_{i\in I} X_i\big) = \{\ (x_i) \in \prod_{i\in I} X_i \ \mid\ \sup_{i\in I} \|x_i\| < \infty\ \}.$$

$\ell_\infty(\prod_{i\in I} X_i)$ is naturally a vector space. An ultrafilter $\mathcal{U}$ on I induces a seminorm on $\ell_\infty(\prod_{i\in I} X_i)$ by defining

$$\|(x_i)\| = \lim_{i,\mathcal{U}} \|x_i\|.$$

The set $N_\mathcal{U}$ of families (x_i) in $\ell_\infty(\prod_{i\in I} X_i)$ such that $\|(x_i)\| = 0$ is obviously a closed subspace of $\ell_\infty(\prod_{i\in I} X_i)$. We define

$$\prod_{i\in I} X_i/\mathcal{U} = \ell_\infty\big(\prod_{i\in I} X_i\big)/N_\mathcal{U}.$$

The space $\prod_{i\in I} X_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultraproduct of* $(X_i)_{i\in I}$. If $X_i = X$ for every $i \in I$, the space $\prod_{i\in I} X_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultrapower of* X and is denoted $X^I/\mathcal{U}$.

If (x_i) is a family in $\ell_\infty(\prod_{i\in I} X_i)$, let us denote by $(x_i)_\mathcal{U}$ the equivalence class of (x_i) in $\prod_{i\in I} X_i/\mathcal{U}$. If $X^I/\mathcal{U}$ is an ultrapower of a normed space X, the map $x \mapsto (x_i)_\mathcal{U}$, where $x_i = x$ for every $i \in I$, is an isometric embedding of X into $X^I/\mathcal{U}$. Hence, we may regard X as a subspace of $X^I/\mathcal{U}$. This embedding is generally not surjective; it is, however, when the ultrafilter $\mathcal{U}$ is principal or the space X is finite dimensional.

1.1. EXERCISE. An ultrafilter $\mathcal{U}$ is said to be *countably incomplete* if there exists sets a countable set $\mathcal{S} \subseteq \mathcal{U}$ such that $\bigcap \mathcal{S} = \emptyset$. (For instance, every nonprincipal ultrafilter on $\mathbb{N}$ is countable incomplete.) Prove that if X is a normed space and $\mathcal{U}$ is a countably incomplete ultrafilter on I, then $X^I/\mathcal{U}$ is complete (*i.e.*, every Cauchy sequence in $X^I/\mathcal{U}$ is convergent).

Suppose that X is a Banach space and T is an operator on X. Then T can be extended to an operator $T^I/\mathcal{U}$ on $X^I/\mathcal{U}$ by defining, for $(x_i)_\mathcal{U}$ in $X^I/\mathcal{U}$,

$$T^I\big((x_i)_\mathcal{U}\big) = (\,T(x_i)\,)_\mathcal{U}.$$

Clearly, $\|T^I\| = \|T\|$.

If $(T_j)_{j\in J}$ is a family of operators on X and $(c_k)_{k\in K}$ is a family of elements of X, we will refer to the structure

$$\mathbf{X} = (\ X,\ T_j,\ c_k \ \mid\ j \in J,\ k \in K\ )$$

as a *Banach space structure*. The space X is called the *universe* of the structure. The structure

$$(\ X^I/\mathcal{U},\ T_j^I/\mathcal{U},\ c_k \ \mid\ j \in J,\ k \in K\ )$$

is called the $\mathcal{U}$-ultrapower of $\mathbf{X}$.

Suppose that $(\mathbf{X})_{i\in I}$ is a family of Banach space structures such that the following conditions hold:

(1) There exist sets J, K such that for each $i \in I$

$$\mathbf{X}_i = (\ X_i,\ T_{i,j},\ c_{i,k} \ \mid\ j \in J,\ k \in K\ ).$$

(2) $\sup_{i\in I} \|T_{i,j}\| < \infty$ for every $j \in J$.
(3) $\sup_{i\in I} \|c_{i,k}\| < \infty$ for every $k \in K$.

Then it is natural to define for each $j \in J$ an operator $\prod_{i\in I} T_{i,j}/\mathcal{U}$ on $\prod_{i\in I} X_i/\mathcal{U}$ by letting

$$\prod_{i\in I} T_{i,j}/\mathcal{U}\,\big((x_i)_\mathcal{U}\big)_{i\in I} = (\,T_{i,j}(x_i)\,)_\mathcal{U}.$$

For every $j \in J$ and $k \in K$, we have

$$\|\prod_{i\in I} T_{i,j}/\mathcal{U}\| = \lim_{i,\mathcal{U}} \|T_{i,j}\|, \qquad \|((c_{i,k})_{i\in I})_\mathcal{U}\| = \lim_{i,\mathcal{U}} \|c_{i,k}\|.$$

The structure

$$\Big(\prod_{i\in I} X_i/\mathcal{U},\ \prod_{i\in I} T_{i,j}/\mathcal{U},\ (c_{i,k})_{i\in I} \ \Big|\ j\in J,\ k\in K \Big)$$

is called the $\mathcal{U}$-*ultraproduct of* $(\mathbf{X}_i)_{i\in I}$ and denoted

$$\prod_{i\in I} \mathbf{X}_i/\mathcal{U}.$$

If $\mathbf{X}_i = \mathbf{X}$ for every $i \in I$, the space $\prod_{i\in I} \mathbf{X}_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultrapower of* $\mathbf{X}$ and is denoted $\mathbf{X}^I/\mathcal{U}$.

What is the relation between a Banach space structure and its ultrapowers? In order to answer this question we need to discuss the logic of *positive bounded formulas* and *approximate satisfaction*. The rest of this chapter is devoted to this goal.

2. Syntax: Positive Bounded Formulas

The fundamental distinction between the concept of language in Banach space model theory and the usual concept of language in first-order logic is that a Banach space language is required to come equipped with norm bounds for the constants and operators.

Suppose that X is a Banach space, $(c_k)_{k\in K}$ is a family of elements of X, and $(T_j)_{j\in I}$ is a family of operators on X, and let

$$\mathbf{X} = (\ X,\ T_j,\ c_k \ \mid\ j \in J,\ k \in K\ )$$

be a Banach space structure. A *language* L for $\mathbf{X}$ consists of the following items.

- A syntactic binary function symbol $+$ for the vector space. addition of X and a syntactic symbol 0 for the additive. identity of X.
- For each rational number r, a monadic function symbol (which we denote also by r) for the scalar multiplication by r.
- For each rational number $M > 0$, monadic predicates for the sets

$$\{\, x \in X \mid \|x\| \le M \,\} \quad \text{and} \quad \{\, x \in X \mid \|x\| \ge M \,\}.$$

- A monadic function symbol (an *operator symbol*) for each operator T_j.
- A syntactic symbol (a *constant symbol*) for each element c_k.
- Upper norm bounds for each element c_k and each operator T_j.

We say that $\mathbf{X}$ is a Banach space L-structure, or simply, an L-structure.

1.2. Remark. For every language L, the class of L-structures is closed under ultraproducts. (Notice that the requirement that the language include bounds for each constant and operator symbols is needed for this.)

Fix a language L for a Banach space structure. We now define sets of strings of symbols called the *terms* and *positive bounded formulas* of L. Both definitions are recursive.

A *term* of L (or an *L-term*) is any string of symbols that can be obtained by finitely many applications of the following rules of formation:

- If x is a syntactic variable, then x is an L-term.
- 0 (the syntactic symbol for the additive identity of X) is an L-term.
- If c is a constant symbol of L, then c is an L-term.
- If t_1 and t_2 are L-terms, then $t_1 + t_2$ is an L-term.
- If r is a symbol for scalar multiplication, and t is an L-term, then $r \cdot (t)$ is an L-term.
- If t is an L-term and T is an operator symbol of L, then $T(t)$ is an L-term.

A *positive bounded formula of L* (or a *positive bounded L-formula*) is a string of symbols that can be obtained by finitely many applications of the following rules of formation:

- If t is an L-term and M is a positive rational number, then the expressions
$$\|t\| \leq M, \qquad \|t\| \geq M$$
are positive bounded L-formulas.
- If φ_1 and φ_2 are positive bounded L-formulas, then the expressions
$$(\varphi_1 \wedge \varphi_2), \qquad (\varphi_1 \vee \varphi_2)$$
are positive bounded L-formulas.
- If φ is a positive bounded L-formula, x is a variable, and M is a positive rational number, then the expressions
$$\exists x(\|x\| \leq M \wedge \varphi),$$
$$\forall x(\|x\| \leq M \to \varphi)$$
are positive bounded L-formulas.

Thus, a positive bounded formula is an expression built up from the *atomic formulas*
$$\|t\| \leq M, \qquad \|t\| \geq M$$

(where t is a term of L and $M > 0$) by using the *positive connectives* $\wedge, \vee$ and the *bounded quantifiers*

$$\exists x(\|x\| \leq M \wedge \quad \dots \quad) \quad \text{and} \quad \forall x(\|x\| \leq M \rightarrow \quad \dots \quad)$$

(where $M > 0$).

If t is a term and M_1, M_2 are real numbers, we write $M_1 \leq \|t\| \leq M_2$ as an abbreviation of the positive bounded formula $(M_1 \leq \|t\| \wedge \|t\| \leq M_2)$. Similarly, we write $\|t\| = M$ as an abbreviation of $(M \leq \|t\| \wedge \|t\| \leq M)$. Often, when the context allows it, we omit the outer parentheses in formulas of the form $(\varphi_1 \wedge \varphi_2)$ or $(\varphi_1 \vee \varphi_2)$. Sometimes we also write $\bigwedge_{i=1}^n \varphi_i$ and $\bigvee_{i=1}^n \varphi_i$ as abbreviations of $\varphi_1 \wedge \cdots \wedge \varphi_n$ and $\varphi_1 \vee \cdots \vee \varphi_n$, respectively.

If φ is a positive bounded formula, a *subformula* of φ is any string of consecutive symbols of φ that is a positive bounded formula in its own right.

A variable x is said to occur *free* in a positive bounded formula φ if x occurs in φ and is not under the scope of any of the quantifiers that occur in φ (*i.e.*, there is at least one occurrence of x in φ that is not within any subformula of φ of the form $\exists x\, \psi$ or $\forall x\, \psi$.) A *positive bounded sentence* is a positive bounded formula without free variables. If t is a term and $x_1, \dots, x_n$ are variables, we write $t(x_1, \dots, x_n)$ to indicate that all the variables occurring in t are among $x_1, \dots, x_n$. Similarly, if φ is a positive bounded formula, we write $\varphi(x_1, \dots, x_n)$ to indicate that all the variables that occur free in φ are among $x_1, \dots, x_n$.

3. Semantics: Interpretations

Suppose that

$$\mathbf{X} = (\ X,\ T_j,\ c_k \ \mid\ j \in J,\ k \in K\ )$$

is a Banach space L-structure and $t(x_1, \dots, x_n)$ is an L-term. If $a_1, \dots, a_n$ are elements of X, we denote by

$$t^{\mathbf{X}}[a_1, \dots, a_n],$$

the element of X that results from interpreting the variable x_i as the element a_i, for $i = 1, \dots, n$. The formal definition is by induction on the complexity of t, as follows:

- If $t(x_1, \dots, x_n)$ is the variable x_i ($i = 1, \dots, n$), then $t^{\mathbf{X}}[a_1, \dots, a_n]$ is the element a_i.
- If t is 0, then $t^{\mathbf{X}}[a_1, \dots, a_n]$ is the additive identity of X.
- If $t(x_1, \dots, x_n)$ is the constant symbol c, then $t^{\mathbf{X}}[a_1, \dots, a_n]$ is the interpretation of c in X.

· If $t(x_1, \dots, x_n)$ is $t_1(x_1, \dots, x_n) + t_1(x_1, \dots, x_n)$, where t_1 and t_2 are L-terms, then

$$t^{\mathbf{X}}[a_1, \dots, a_n] = t_1^{\mathbf{X}}[a_1, \dots, a_n] + t_1^{\mathbf{X}}[a_1, \dots, a_n].$$

· If $t(x_1, \dots, x_n)$ is $r \cdot (t_1(x_1, \dots, x_n))$, where t_1 is an L-term and r is a syntactic symbol for scalar multiplication, then

$$t^{\mathbf{X}}[a_1, \dots, a_n] = r(t^{\mathbf{X}}[a_1, \dots, a_n]).$$

· If $t(x_1, \dots, x_n)$ is $T(t_1(x_1, \dots, x_n))$, where where T is a syntactic symbol for the operator T_j and t_1 is an L-term, then

$$t^{\mathbf{X}}[a_1, \dots, a_n] = T_j(t_1^{\mathbf{X}}[a_1, \dots, a_n]).$$

Note that $t^{\mathbf{X}}[a_1, \dots, a_n]$ depends not only on t, $\mathbf{X}$, and $a_1, \dots, a_n$, but also on a given list of variables that is not given explicitly by the notation. The context will normally make it it clear which list of variables is being considered.

If $\varphi(x_1, \dots, x_n)$ is an L-formula and $a_1, \dots, a_n$ are elements of X, we write

$$\mathbf{X} \models \varphi[a_1, \dots, a_n]$$

if the formula φ is true in $\mathbf{X}$ when the variable x_i is interpreted as the element a_i, for $i = 1, \dots, n$. This concept should be intuitively clear. The formal definition is by induction on the complexity of φ, as follows:

· If φ is $t(x_1, \dots, x_n) \leq M$, $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ if and only if

$$t^{\mathbf{X}}[a_1, \dots, a_n] \leq M.$$

· If φ is $t(x_1, \dots, x_n) \geq M$, $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ if and only if

$$t^{\mathbf{X}}[a_1, \dots, a_n] \geq M.$$

· If φ is $(\psi_1 \wedge \psi_2)$, then $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ if and only if

$$\mathbf{X} \models \psi_1[a_1, \dots, a_n] \quad \text{and} \quad \mathbf{X} \models \psi_2[a_1, \dots, a_n].$$

· If φ is $(\psi_1 \vee \psi_2)$, then $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ if and only if

$$\mathbf{X} \models \psi_1[a_1, \dots, a_n] \quad \text{or} \quad \mathbf{X} \models \psi_2[a_1, \dots, a_n].$$

· If φ is $\exists x(\|x\| \leq M \wedge \psi(x, x_1, \dots, x_n))$, where M is a positive rational number and x is a variable, then $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ if and only if

$$\mathbf{X} \models \psi[a, a_1, \dots, a_n], \quad \text{for some } a \in X \text{ with } \|a\| \leq M.$$

· If φ is $\forall x(\|x\| \leq M \to \psi(x, x_1, \dots, x_n))$, where M is a positive rational number and x is a variable, then $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ if and only if

$$\mathbf{X} \models \psi[a, a_1, \dots, a_n], \quad \text{for every } a \in X \text{ with } \|a\| \leq M.$$

If Γ is a set of positive bounded formulas, we write $\Gamma(x_1, \dots, x_n)$ to indicate that all the variables that occur free in formulas of Γ are among $x_1, \dots, x_n$. If $\mathbf{X}$ is a Banach space L-structure and $\Gamma(x_1, \dots, x_n)$ is a set of positive bounded formulas, we write

$$\mathbf{X} \models \Gamma[a_1, \dots, a_n]$$

if $\mathbf{X} \models \varphi[a_1, \dots, a_n]$ for every formula $\varphi(x_1, \dots, x_n) \in \Gamma(x_1, \dots, x_n)$.

4. Approximations of Formulas

If φ is a positive bounded formula, an approximation φ' of φ is a positive bounded formula that results from relaxing all the norm estimates in φ. We indicate that φ' is an approximation of φ by writing $\varphi < \varphi'$ (or equivalently $\varphi' > \varphi$). The formal definition is by induction on the complexity of φ and is given by the following table.

If φ is:	The approximations of φ are:
$M \leq \|t\|$	$M' \leq \|t\|$, where $M' < M$
$\|t\| \leq M$	$\|t\| \leq M'$, where $M' > M$
$(\psi_1 \wedge \psi_2)$	$(\psi_1' \wedge \psi_2')$, where $\psi_i' > \psi_i$, for $i = 1, 2$
$(\psi_1 \vee \psi_2)$	$(\psi_1' \vee \psi_2')$, where $\psi_i' > \psi_i$, for $i = 1, 2$
$\exists x(\|x\| \leq M \wedge \psi)$	$\exists x(\|x\| \leq M' \wedge \psi')$, where $M' > M$ and $\psi' > \psi$
$\forall x(\|x\| \leq M \to \psi)$	$\forall x(\|x\| \leq M' \to \psi')$, where $M' < M$ and $\psi' > \psi$

1.3. Remark. Suppose that $\mathbf{X}$ is a Banach space L-structure, $a_1, \dots, a_n$ are elements of the universe of $\mathbf{X}$, and $\varphi(x_1, \dots, x_n)$ is a positive bounded L-formula. Then

$$\mathbf{X} \models \varphi[a_1, \dots, a_n] \quad \text{implies} \quad \mathbf{X} \models \varphi'[a_1, \dots, a_n], \quad \text{for every } \varphi' > \varphi.$$

1.4. Notation. If Γ is a set of positive bounded formulas, we denote by Γ_+ the set of all approximations of formulas in Γ.

The negation connective is not allowed in positive bounded formulas, nor is the implication connective, except when it occurs as part of the bounded universal quantifiers. However, for every positive bounded formula φ there is a positive bounded formula $\operatorname{neg}(\varphi)$ (the *weak negation of* φ) that in Banach space model theory plays a role analogous to that played by the negation of φ in traditional model theory. The connective neg is defined recursively as follows.

If φ is:	$\operatorname{neg}(\varphi)$ is:
$\|t\| \leq M$	$\|t\| \geq M$
$\|t\| \geq M$	$\|t\| \leq M$
$(\psi_1 \wedge \psi_2)$	$\operatorname{neg}(\psi_1) \vee \operatorname{neg}(\psi_2)$
$(\psi_1 \vee \psi_2)$	$\operatorname{neg}(\psi_1) \wedge \operatorname{neg}(\psi_2)$
$\exists x(\|x\| \leq M \wedge \psi)$	$\forall x(\|x\| \leq M \rightarrow \operatorname{neg}(\psi))$
$\forall x(\|x\| \leq M \rightarrow \psi)$	$\exists x(\|x\| \leq M \wedge \operatorname{neg}(\psi))$

1.5. Remarks.

(1) If φ, φ' are positive bounded formulas, then $\varphi < \varphi'$ if and only if $\operatorname{neg}(\varphi') < \operatorname{neg}(\varphi)$.

(2) If $\mathbf{X}$ is a Banach space L-structure and $\varphi(x_1, \ldots, x_n)$ is a positive bounded L-formula such that, $\mathbf{X} \not\models \varphi[a_1, \ldots, a_n]$, then $\mathbf{X} \models \operatorname{neg}(\varphi)[a_1, \ldots, a_n]$. If φ' is an approximation of φ such that $\mathbf{X} \models \operatorname{neg}(\varphi')[a_1, \ldots, a_n]$, then $\mathbf{X} \not\models \varphi[a_1, \ldots, a_n]$.

1.6. Proposition (Perturbation Lemma). *For every positive bounded L-formula $\varphi(x_1, \ldots, x_n)$, every $\varphi' > \varphi$, and every $M > 0$ there exists $\delta > 0$ such that the following condition holds. If $\mathbf{X}$ is a Banach space L-structure and $a_1, \ldots, a_n$ are elements of the universe of $\mathbf{X}$ such that*

$$\mathbf{X} \models \bigwedge_{1 \leq i \leq n} \|a_i\| \leq M \ \wedge \ \varphi[a_1, \ldots, a_n],$$

then whenever $b_1, \ldots, b_n$ are elements of the universe of $\mathbf{X}$ satisfying

$$\max_{1 \leq i \leq n} \|a_i - b_i\| < \delta,$$

we have

$$\mathbf{X} \models \varphi'[b_1, \ldots, b_n].$$

Sketch of proof. By induction on the complexity of φ, using the fact that both the norm and the operations of $\mathbf{X}$ are uniformly continuous on every bounded subset of the universe $\mathbf{X}$ (and the moduli of uniform continuity are given by the language L, so they do not depend on the structure $\mathbf{X}$). $\square$

5. Approximate Satisfaction

Suppose that $\mathbf{X}$ is a Banach space L-structure, $\varphi(x_1,\dots,x_n)$ is a positive bounded L-formula, and $a_1,\dots,a_n$ are elements of the universe of $\mathbf{X}$. We say that $\mathbf{X}$ *approximately satisfies* $\varphi[a_1,\dots,a_n]$, and write

$$\mathbf{X} \models_{\mathcal{A}} \varphi[a_1,\dots,a_n],$$

if

$$\mathbf{X} \models \varphi'[a_1,\dots,a_n], \qquad \text{for every approximation } \varphi' \text{ of } \varphi.$$

If $\Gamma(x_1,\dots,x_n)$ is a set of positive bounded formulas, we say that $\mathbf{X}$ *approximately satisfies* $\Gamma[a_1,\dots,a_n]$, and write

$$\mathbf{X} \models_{\mathcal{A}} \Gamma[a_1,\dots,a_n],$$

if $\mathbf{X} \models_{\mathcal{A}} \varphi[a_1,\dots,a_n]$ for every formula $\varphi \in \Gamma$. In the notation introduced in 1.4,

$$\mathbf{X} \models_{\mathcal{A}} \Gamma[a_1,\dots,a_n] \quad \text{if and only if} \quad \mathbf{X} \models \Gamma_+[a_1,\dots,a_n].$$

The notion of approximate satisfaction, rather than the usual notion of satisfaction, provides the appropriate semantics for a model-theoretic analysis of Banach space structures.

A *quantifier-free formula* is a formula that does not include quantifiers.

1.7. Remark. For quantifier-free positive bounded formulas, the concepts of $\models$ and $\models_{\mathcal{A}}$ are equivalent. However, for general formulas, $\models_{\mathcal{A}}$ is strictly weaker than $\models$. To see this, let $\ell_p(n)$ denote the space $\mathbb{R}^n$ equipped with the ℓ_p-norm. Consider the sentence

$$\varphi : \exists x \exists y \big(\, \|x\| = 1 \wedge \|y\| = 1 \wedge \|x+y\| = 1 \wedge \|x-y\| = 1 \, \big).$$

Then, if X is a Banach space, $X \models \varphi$ if and only if X contains a 2-dimensional subspace isometric to $\ell_\infty(2)$. Take a sequence (p_n) of real numbers such that $1 \le p_0 < p_1 < \dots$ and $p_n \to \infty$ as $n \to \infty$, and let X be an ℓ_2-sum of the spaces $\ell_{p_n}(2)$, for $n \in \mathbb{N}$. Then $X \models_{\mathcal{A}} \varphi$, but $X \not\models \varphi$.

The class of positive bounded formulas is not closed under negations. However, as the following proposition shows, weak negations are sufficient to express the fact that a formula is not approximately satisfied in a structure.

1.8. PROPOSITION. *Suppose that* $\mathbf{X}$ *is a Banach space* L*-structure and* $a_1, \ldots, a_n$ *are elements of the universe of* $\mathbf{X}$*. Then, for every positive bounded* L*-formula* $\varphi(x_1, \ldots, x_n)$ *, we have* $\mathbf{X} \not\models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$ *if and only there exists a formula* $\varphi' > \varphi$ *such that* $\mathbf{X} \models_{\mathcal{A}} \mathrm{neg}(\varphi')[a_1, \ldots, a_n]$.

PROOF. In order to simplify the nomenclature, let us suppress the lists $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ from the notation.

If $\mathbf{X} \not\models_{\mathcal{A}} \varphi$, there exists $\varphi' > \psi$ such that $\mathbf{X} \not\models \varphi'$. Then $\mathbf{X} \models \mathrm{neg}(\varphi')$ and hence $\mathbf{X} \models_{\mathcal{A}} \mathrm{neg}(\varphi')$. Conversely, assume that there exists $\varphi' > \varphi$ such that $\mathbf{X} \models_{\mathcal{A}} \mathrm{neg}(\varphi')$ and take sentences ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. Then $\mathbf{X} \models \mathrm{neg}(\psi')$ (by Remark 1.5) and hence $\mathbf{X} \not\models \psi$, so $\mathbf{X} \not\models_{\mathcal{A}} \varphi$. □

6. Beginning Model Theory

The following theorem establishes the key connection between ultraproducts and approximate satisfaction:

1.9. THEOREM. *Let* $(\mathbf{X}_i)_{i \in I}$ *is a family of Banach space* L*-structures, let* $\mathcal{U}$ *be an ultrafilter on* I*, and for each* $i \in I$ *let* π_i *denote the natural projection from* $\prod_{i \in I} \mathbf{X}_i / \mathcal{U}$ *onto* $\mathbf{X}_i$*. Let* $\varphi(x_1, \ldots, x_n)$ *be a positive bounded* L*-formula and let* $a_1, \ldots, a_n$ *be elements of the universe of* $\prod_{i \in I} \mathbf{X}_i / \mathcal{U}$*. Then,*

$$\prod_{i \in I} \mathbf{X}_i / \mathcal{U} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$$

if and only if for every approximation φ' *of* φ*, the set*

$$\{\, i \in I \mid \mathbf{X}_i \models \varphi'[\pi_i(a_1), \ldots, \pi_i(a_n)] \,\}$$

is in $\mathcal{U}$*.*

PROOF. By induction on the complexity of φ. (Use Remark 1.5.) □

From this, we obtain the compactness theorem, which is the cornerstone of the (first-order) model theory of Banach space structures:

1.10. THEOREM (Compactness). *Let* Γ *be a set of positive bounded* L*-sentences such that every finite subset of* Γ *is approximately satisfied by some Banach space* L*-structure. Then there exists a Banach space* L*-structure that approximately satisfies every sentence in* Γ*.*

SKETCH OF PROOF. Let I be the set of finite subsets of Γ_+, and for each $i \in I$ let $\mathbf{X}_i$ be a Banach space L-structure satisfying every sentence in i. For every finite subset Δ of Γ_+ let F_Δ be the set of all $i \in I$ such that $\mathbf{X}_i \models \Delta$. The family $\mathcal{F}$ of sets of the form F_Δ is closed under finite intersections. If $\mathcal{U}$ is an ultrafilter on I extending $\mathcal{F}$, we have

$$\prod_{i \in I} \mathbf{X}_i / \mathcal{U} \models_{\mathcal{A}} \Gamma.$$

$\square$

A *positive bounded theory* is a set of positive bounded sentences. If $\mathbf{X}$ is a Banach space structure, we denote by $\mathrm{Th}_{\mathcal{A}}(\mathbf{X})$ the set of sentences that are approximately satisfied by $\mathbf{X}$.

1.11. COROLLARY. *The following conditions are equivalent for a positive bounded theory Γ in a language L.*

(1) *There exists a Banach space L-structure $\mathbf{X}$ such that $\Gamma = \mathrm{Th}_{\mathcal{A}}(\mathbf{X})$.*
(2) (a) *Every finite subset of Γ is approximately satisfied in some Banach space L-structure,*
    (b) *For every positive bounded L-sentence φ, either $\varphi \in \Gamma$ or there exists $\varphi' > \varphi$ such that $\mathrm{neg}(\varphi') \in \Gamma$.*

PROOF. The implication (1) $\Rightarrow$ (2) follows from Proposition 1.8. To prove (2) $\Rightarrow$ (1), use Theorem 1.10 to fix a Banach space L-structure $\mathbf{X}$ such that $\mathbf{X} \models_{\mathcal{A}} \Gamma$. Then $\mathrm{Th}_{\mathcal{A}}(\mathbf{X}) \subseteq \Gamma$, for if φ were in $\mathrm{Th}_{\mathcal{A}}(\mathbf{X}) \setminus \Gamma$, there would exist $\varphi' > \varphi$ such that $\mathrm{neg}(\varphi') \in \Gamma \subseteq \mathrm{Th}_{\mathcal{A}}(\mathbf{X})$, which is impossible. Hence $\Gamma = \mathrm{Th}_{\mathcal{A}}(\mathbf{X})$. $\square$

If $\mathbf{X}$ and $\mathbf{Y}$ are Banach space L-structures, we say that $\mathbf{X}$ and $\mathbf{Y}$ are *approximately elementarily equivalent*, and write

$$\mathbf{X} \equiv_{\mathcal{A}} \mathbf{Y},$$

if $\mathbf{X}$ and $\mathbf{Y}$ approximately satisfy the same positive bounded L-sentences.

Suppose that

$$\mathbf{X} = (\ X,\ T_j,\ c_k \ \mid\ j \in J,\ k \in K\ ),$$
$$\mathbf{Y} = (\ Y,\ U_j,\ c_k \ \mid\ j \in J,\ k \in K\ )$$

are Banach space structures. We will say that $\mathbf{X}$ is a *substructure* of $\mathbf{Y}$ if X is a subspace of Y, $c_k \in X$ for every $k \in K$, and U_j extends T_j, for every $j \in J$.

If $\mathbf{X}$ is as above and $(d_l)_{l \in L}$ is a family of elements of X, we sometimes denote the structure

$$(\ X,\ T_j,\ c_k,\ d_l,\ \mid\ j \in J,\ k \in K,\ l \in L\ )$$

as

$$(\mathbf{X}, d_l \mid l \in L).$$

Such a structure is called an *expansion of* $\mathbf{X}$ *by constants.*

If $\mathbf{X}$ is a substructure of $\mathbf{Y}$, we say that $\mathbf{X}$ is an *approximately elementary substructure of* $\mathbf{Y}$, and write

$$\mathbf{X} \prec_{\mathcal{A}} \mathbf{Y},$$

(or equivalently $\mathbf{Y} \succ_{\mathcal{A}} \mathbf{X}$) if

$$(\,\mathbf{X},\, a \mid a \in X\,) \equiv_{\mathcal{A}} (\,\mathbf{Y},\, a \mid a \in X\,).$$

We also say the $\mathbf{Y}$ is an *approximately elementary extension* of $\mathbf{X}$.

1.12. Proposition. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are* L*-structures and the universe of* $\mathbf{X}$ *is* X*.*

(1) *If* A *is a subset of* X *and* A_0 *is a dense subset of* A*, then*

$$(\,\mathbf{X},\, a \mid a \in A_0\,) \equiv_{\mathcal{A}} (\,Y,\, a \mid a \in A_0\,)$$

implies

$$(\,\mathbf{X},\, a \mid a \in A\,) \equiv_{\mathcal{A}} (\,\mathbf{Y},\, a \mid a \in A\,).$$

(2) (Tarski-Vaught Test.) *If* $\mathbf{X}$ *is an* L*-substructure of* $\mathbf{Y}$*, then* $\mathbf{X} \prec_{\mathcal{A}} \mathbf{Y}$ *if and only if the following condition holds: For every positive bounded sentence* φ *in a language for* $(\,\mathbf{Y},\, a \mid a \in X\,)$ *of the form* $\exists x(\psi(x))$ *such that* $\mathbf{Y} \models_{\mathcal{A}} \varphi$ *and every approximation* ψ' *of* ψ *there exists* $a \in X$ *such that* $\mathbf{Y} \models_{\mathcal{A}} \psi'[a]$*.*

Sketch of proof. Part (1) of the proposition follows from the Perturbation Lemma (Proposition 1.6). Part (2) is proved by induction on formulas. □

Suppose that

$$\mathbf{X} = (\;X,\; T_j,\; c_k \;\mid\; j \in J,\; k \in K\;),$$
$$\mathbf{Y} = (\;Y,\; U_j,\; d_k \;\mid\; j \in J,\; k \in K\;).$$

An *embedding* of $\mathbf{X}$ into $\mathbf{Y}$ is an isometric isomorphism $f\colon X \to Y$ such that the structure

$$f(\mathbf{X}) = (\;f(X),\; f(T_j),\; f(c_k) \;\mid\; j \in J,\; k \in K\;),$$

where $f(T_j)$ is the operator on $f(X)$ defined by $f(T_j)(f(x)) = f(T_j(x))$, is a substructure of $\mathbf{Y}$.

1.13. Proposition. *Let* $\mathbf{X}$ *be a Banach space structure.*

(1) *If* $\hat{\mathbf{X}}$ *is an ultrapower of* $\mathbf{X}$*, then* $\mathbf{X} \prec_{\mathcal{A}} \hat{\mathbf{X}}$*.*

(2) *If* $\mathbf{Y}$ *is a Banach space structure, then* $\mathbf{Y} \equiv_{\mathcal{A}} \mathbf{X}$ *if and only if there exists a Banach space structure* $\hat{\mathbf{X}} \succ_{\mathcal{A}} \mathbf{X}$ *and an embedding* $f : \mathbf{Y} \to \hat{\mathbf{X}}$ *such that* $f(\mathbf{Y}) \prec_{\mathcal{A}} \hat{\mathbf{X}}$. *Furthermore,* $\hat{\mathbf{X}}$ *can be taken to be an ultrapower of* $\mathbf{X}$.

Sketch of proof. (1) follows from the Tarski-Vaught Test (see Proposition 1.12), or directly from Theorem 1.9. The implication $\Leftarrow$ of (2) follows from (1). To prove the implication $\Rightarrow$ of (2), let Y be the universe of $\mathbf{Y}$, let L' be an expansion of L that contains constant symbols for all the elements of Y, and define

$$\Gamma = \mathrm{Th}_{\mathcal{A}}(\mathbf{Y}, a \mid a \in Y).$$

Notice that since $\mathbf{Y} \equiv_{\mathcal{A}} \mathbf{X}$, every finite subset of Γ approximately satisfied by an expansion of $\mathbf{X}$ by constants, namely, an expansion of $\mathbf{X}$ to an L'-structure. Therefore, arguing as in the proof of the Compactness Theorem (Theorem 1.10) one finds an ultrapower $\hat{\mathbf{X}}$ of $\mathbf{X}$ and an expansion of $\hat{\mathbf{X}}$ to an L'-structure that approximately satisfies Γ. □

Recall that the *density* (or density character) of a topological space is the smallest cardinality of a dense subset of the space. For example, a space is separable if and only if its density is $\aleph_0$.

1.14. Proposition. *Suppose that* L *is countable and* $\mathbf{X}$ *is a Banach space* L*-structure with universe* X.

(1) (Downward Löwenheim-Skolem Theorem.) *For every set* $A \subseteq X$ *there exists a substructure* $\mathbf{Y}$ *of* $\mathbf{X}$ *with universe* Y *such that* $A \subseteq Y$,

$$\mathrm{density}(Y) = \mathrm{density}(A),$$

and $\mathbf{Y} \prec_{\mathcal{A}} \mathbf{X}$.

(2) (Upward Löwenheim-Skolem Theorem.) *If* X *is infinite-dimensional, then for every cardinal* κ *with* $\kappa \geq \mathrm{density}(X)$ *there exists an approximately elementary extension of* X *of density* κ.

Sketch of proof. To prove (1), let A_0 be a dense subset of A and expand the language with constant symbols and norm bounds for the elements of A_0. Now apply Proposition 1.12 to the structure $(\, X,\, a \mid a \in A_0\,)$.

To prove (2), let X_0 be a dense subset of X and expand the language with constants symbols and norm bounds for the elements of X_0. Expand the language further with new constants symbols $\{\, c_i \,\}_{i<\kappa}$ and norm bounds $\|c_i\| = 1$ for $i < \kappa$. Every finite subset of the theory

$$\mathrm{Th}_{\mathcal{A}}(\, \mathbf{X},\, a \mid a \in X_0\,) \cup \{\; \|c_i - c_j\| = 1 \;\mid\; i < j < \kappa \;\}.$$

is approximately satisfied in $\mathbf{X}$, so the conclusion now follows from (1). □

7. $(1+\epsilon)$-Isomorphism and $(1+\epsilon)$-Equivalence of Structures

We now address the question of when two Banach spaces have isomorphic approximately elementary extensions.

In the following discussion, L_0 will denote a language that contains no operator symbols.

For every L_0-formula φ and every rational $\epsilon > 0$ we define an approximation $\varphi_{1+\epsilon}$ as follows.

In φ:	In $\varphi_{1+\epsilon}$:
$\|t\| \le M$	$\|t\| \le M(1+\epsilon)$
$\|t\| \ge M$	$\|t\| \ge \frac{M}{1+\epsilon}$
$\psi_1 \wedge \psi_2$	$(\psi_1)_{1+\epsilon} \wedge (\psi_2)_{1+\epsilon}$
$\psi_1 \vee \psi_2$	$(\psi_1)_{1+\epsilon} \vee (\psi_2)_{1+\epsilon}$
$\exists x(\ \|x\| \le M \wedge \psi\ )$	$\exists x(\ \|x\| \le M(1+\epsilon) \wedge \psi_{1+\epsilon}\ )$
$\forall x(\ \|x\| \le M \to \psi\ )$	$\forall x(\ \|x\| \le \frac{M}{1+\epsilon} \to \psi_{1+\epsilon}\ )$

If Γ is a set of L_0-formulas, we denote by $\Gamma_{1+\epsilon}$ the set of $(1+\epsilon)$-approximations of formulas in Γ.

We say that two Banach space L_0-structures $\mathbf{X}$ and $\mathbf{Y}$ are $(1+\epsilon)$-*equivalent*, and write

$$\mathbf{X} \equiv_{1+\epsilon} \mathbf{Y},$$

if for every L_0-sentence φ,

$$\mathbf{X} \models_{\mathcal{A}} \varphi \qquad \text{implies} \qquad \mathbf{Y} \models_{\mathcal{A}} \varphi_{1+\epsilon}.$$

Let us prove that $\equiv_{1+\epsilon}$ is a symmetric relation. Suppose

$$(\,\mathrm{Th}_{\mathcal{A}}(\mathbf{X})\,)_{1+\epsilon} \subseteq \mathrm{Th}_{\mathcal{A}}(\mathbf{Y}),$$

take a positive bounded sentence φ such that $\mathbf{Y} \models_{\mathcal{A}} \varphi$, and fix $\theta > \varphi_{1+\epsilon}$ in order to show that $\mathbf{X} \models \theta$. Choose $\varphi' > \varphi$ such that $\varphi_{1+\epsilon} < \varphi'_{1+\epsilon} < \theta$. If $\mathbf{X} \not\models \theta$, then $\mathbf{X} \not\models \varphi'_{1+\epsilon}$, so $\mathbf{X} \models \mathrm{neg}(\varphi'_{1+\epsilon})$. By assumption, $\mathbf{Y} \models_{\mathcal{A}} (\,\mathrm{neg}(\varphi'_{1+\epsilon})\,)_{1+\epsilon}$. But $(\,\mathrm{neg}(\varphi'_{1+\epsilon})\,)_{1+\epsilon}$ is equivalent to $\mathrm{neg}(\varphi')$, so $\mathbf{Y} \models_{\mathcal{A}} \mathrm{neg}(\varphi')$. This contradicts the choice of φ, by Proposition 1.8.

If $\epsilon > 0$, two structures

$$(\ X,\ c_i \ \mid\ i \in I\ )$$

and

$$(\ Y,\ d_i \ \mid\ i \in I\ )$$

are said to be $(1+\epsilon)$-*isomorphic* if there exists a linear isomorphism $f\colon X \to Y$ such that $f(c_i) = d_i$ for every $i \in I$ and $\|f\|, \|f^{-1}\| \le 1+\epsilon$, *i.e.*,

$$(1+\epsilon)^{-1}\|x\| \le \|f(x)\| \le (1+\epsilon)\|x\|$$

for every $x \in X$. The function f is called a $(1+\epsilon)$-*isomorphism.*

It is easy to see that two $(1+\epsilon)$-isomorphic structures are $(1+\epsilon)$-equivalent. The following is a converse of this observation.

1.15. THEOREM. *Two Banach space L_0-structures are $(1+\epsilon)$-equivalent if and only if they have $(1+\epsilon)$-isomorphic approximately elementary extensions.*

SKETCH OF PROOF. We prove the nontrivial implication. Suppose

$$\begin{aligned}\mathbf{X} &= (\ X,\ c_i \ \mid\ i \in I\ ),\\ \mathbf{Y} &= (\ Y,\ d_i \ \mid\ i \in I\ )\end{aligned}$$

and assume $\mathbf{X} \equiv_{1+\epsilon} \mathbf{Y}$. Using compactness (Theorem 1.10), we construct chains of extensions

$$\begin{aligned}\mathbf{X} = \mathbf{X}_0 \prec_{\mathcal{A}} \mathbf{X}_1 \prec_{\mathcal{A}} \mathbf{X}_2 \prec_{\mathcal{A}} \cdots\\ \mathbf{Y} = \mathbf{Y}_0 \prec_{\mathcal{A}} \mathbf{Y}_1 \prec_{\mathcal{A}} \mathbf{Y}_2 \prec_{\mathcal{A}} \cdots\end{aligned}$$

and embeddings

$$\begin{tikzcd}
\mathbf{X}_0 \arrow[r, phantom, "\prec_{\mathcal{A}}"] \arrow[dr, hook, "f_1"] & \mathbf{X}_1 \arrow[r, phantom, "\prec_{\mathcal{A}}"] \arrow[dr, hook, "f_2"] & \mathbf{X}_2 \arrow[r, phantom, "\prec_{\mathcal{A}}"] & \cdots \\
\mathbf{Y}_0 \arrow[r, phantom, "\prec_{\mathcal{A}}"] & \mathbf{Y}_1 \arrow[r, phantom, "\prec_{\mathcal{A}}"] \arrow[u, hook, "g_1"'] & \mathbf{Y}_2 \arrow[r, phantom, "\prec_{\mathcal{A}}"] \arrow[u, hook, "g_2"'] & \cdots
\end{tikzcd}$$

such that

$$f_n \subseteq g_n^{-1} \subseteq f_{n+1}, \qquad \text{for } n = 1, 2, \ldots$$

and for every quantifier-free formula $\varphi(\bar{x})$,

$$\mathbf{X}_n \models \varphi[\bar{a}] \qquad \text{implies} \qquad \mathbf{Y}_{n+1} \models \varphi_{1+\epsilon}[\,f_{n+1}(\bar{a})\,]$$

and

$$\mathbf{Y}_n \models \varphi[\bar{a}] \qquad \text{implies} \qquad \mathbf{X}_n \models \varphi_{1+\epsilon}[\,g_n(\bar{a})\,].$$

Let X_n and Y_n be the universes of $\mathbf{X}_n$ and $\mathbf{Y}_n$, respectively, and let $\hat{X}$ and $\hat{Y}$ denote the norm-completions of $\bigcup_{n>0} X_n$ and $\bigcup_{n>0} Y_n$, respectively. Then $\bigcup_{n>0} f_n$ extends naturally to a $(1+\epsilon)$-isomorphism $f\colon \hat{X} \to \hat{Y}$, and

$\bigcup_{n>0} g_n$ extends to a $(1+\epsilon)$-isomorphism $g\colon \hat{Y} \to \hat{X}$ such that $g = f^{-1}$. Consider the L_0-structures

$$\hat{\mathbf{X}} = (\ \hat{X},\ c_i \ \mid\ i \in I\ ),$$
$$\hat{\mathbf{Y}} = (\ \hat{Y},\ d_i \ \mid\ i \in I\ )$$

Then $\mathbf{X} \prec_{\mathcal{A}} \hat{\mathbf{X}}$, $\mathbf{Y} \prec_{\mathcal{A}} \hat{\mathbf{Y}}$, and f is a $(1+\epsilon)$-isomorphism between $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$. $\square$

8. Finite Representability

The notion of finite representability is the central notion in local Banach space geometry.

A Banach space X is *finitely representable* in a Banach space Y if for every finite dimensional subspace E of X and for every $\epsilon > 0$ there exists a finite dimensional subspace F of Y such that E and F are $(1+\epsilon)$-isomorphic.

If $\mathbf{X}$ is a Banach space structure, the *existential theory of* $\mathbf{X}$, denoted $\exists\,\mathrm{Th}_{\mathcal{A}}(\mathbf{X})$ is the set of existential positive bounded sentences that are approximately satisfied by $\mathbf{X}$.

1.16. Proposition. *Let X and Y be Banach spaces. The following conditions are equivalent.*

(1) *X is finitely representable in Y.*
(2) *$\exists\,\mathrm{Th}_{\mathcal{A}}(X) \subseteq \mathrm{Th}_{\mathcal{A}}(Y)$.*
(3) *There exists an ultrapower of Y that contains an isometric copy of X.*

Sketch of proof. The implication (3) $\Rightarrow$ (1) is immediate, since an ultrapower of Y is always finitely representable in Y. The implication (1) $\Rightarrow$ (2) follows from the fact that the unit ball of a finite dimensional space is compact. To prove (2) $\Rightarrow$ (3), assume that X is finitely representable in Y and let Γ be set of all quantifier-free sentences that are satisfied by the structure $(X, a \mid a \in X)$. By compactness (Theorem 1.10), there is an ultrapower $\hat{Y}$ of Y such that $\hat{Y} \models_{\mathcal{A}} \Gamma$. Since $\models_{\mathcal{A}}$ and $\models$ coincide for quantifier-free formulas, we have $\hat{Y} \models \Gamma$, so $\hat{Y}$ contains an isometric copy of X. $\square$

9. Types

Suppose that $\mathbf{X}$ is a Banach a space structure with universe X. If $\bar{c}$ is a finite tuple of elements of X and A is a subset of X, the *type of $\bar{c}$ over A* is the set of positive bounded formulas

$$\mathrm{tp}(\bar{c}/A) = \{\ \varphi(\bar{x}, \bar{a})\ \mid \bar{a} \in A,\ (\,\mathbf{X},\, a \mid a \in A\,) \models_{\mathcal{A}} \varphi(\bar{c}, \bar{a})\ \}.$$

1.17. Proposition. *Let* $\mathbf{X}$ *be a Banach space structure, let* A *be a subset of the universe of* $\mathbf{X}$*, and let* L *be a language for the structure* $(\,\mathbf{X}, a \mid a \in A\,)$*. The following conditions are equivalent for a set of positive bounded* L*-formulas* $t(\bar{x}) = t(x_1, \ldots, x_n)$*.*

(1) *There exists a Banach space structure* $\mathbf{Y} \succ_{\mathcal{A}} \mathbf{X}$ *and an* n*-tuple* $\bar{c}$ *of elements of the universe of* $\mathbf{Y}$ *such that* $t(\bar{x}) = \mathrm{tp}(\bar{c}/A)$*.*

(2) (a) *There exists* $M > 0$ *such that the formula*

$$\bigwedge_{1 \le i \le n} \|x_i\| \le M$$

is in t*,*

(b) *Every formula in* t_+ *is satisfied in* $(\,\mathbf{X}, a \mid a \in A\,)$*,*

(c) *For every* L*-formula* $\varphi(\bar{x})$*, either* $\varphi \in t$*, or there exists* $\varphi' > \varphi$ *such that* $\mathrm{neg}(\varphi') \in t$*.*

Sketch of proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 1.8. The implication (2) $\Rightarrow$ (1) follows from compactness (Theorem 1.10) and, again, Proposition 1.8. $\square$

1.18. Remark. Condition 2-(c) in Proposition 1.17 can be replaced by the following equivalent condition:

$$\varphi(\bar{x}) \in t \qquad \text{if and only if} \qquad \varphi(x)_+ \subseteq t,$$

where $\varphi(x)_+$ denotes the set of all approximations of φ.

If $\mathbf{X}$ is a Banach space structure, A is a subset of the universe of $\mathbf{X}$, and $t(\bar{x})$ is a set of positive bounded formulas satisfying the equivalent conditions of Proposition 1.17, we say that t is a *type over* A, and that $\bar{c}$ *realizes* t (or $\bar{c}$ is a *realization* of t) in $\mathbf{Y}$. If $\bar{x} = x_1, \ldots, x_n$, we call t an n*-type*.

Fix a Banach space structure $\mathbf{X}$, a subset A of the universe of $\mathbf{X}$, and a language L for $(\,X, a \mid a \in A\,)$. Given a positive bounded L-formula φ, let $[\varphi]$ denote the set of types over A that contain φ. The *logic topology* is the topology on the set of types over A where the basic open neighborhoods of a type t are the sets of the form $[\varphi]$, with $\varphi \in t_+$. (These sets form a basis for a topology since t_+ is closed under finite conjunctions.) The logic topology is Hausdorff.

If $t(x_1, \ldots, x_n)$ is a type and $(c_1, \ldots, c_n)$ is a realization of t, we define the *norm* of t, denoted $\|t\|$, as the number $\max_{1 \le i \le n} \|c_i\|$. Notice that the norm $\|t\|$ depends only on t and not on the particular realization used to compute it.

1.19. PROPOSITION. *For any $M > 0$, the set of types of norm less than or equal to M is compact with respect to the logic topology.*

SKETCH OF PROOF. Fix a Banach space structure $\mathbf{X}$ and a subset A of the universe of $\mathbf{X}$. Let $(t_i)_{i\in I}$ be a family of types over A and let $\mathcal{U}$ be an ultrafilter on I. By compactness (Theorem 1.10), for each i we can fix a Banach space structure $\mathbf{Y}_i \succ_{\mathcal{A}} \mathbf{X}$ such that t_i is realized in $\mathbf{Y}_i$. For each $i \in I$ let $\bar{c}_i$ be a realization of t_i in $\mathbf{Y}_i$. It is now easy to see that the type over A of the element of $\prod_{i\in I} \mathbf{Y}_i/\mathcal{U}$ represented by $(\bar{c}_i)_{i\in I}$ is $\lim_{i,\mathcal{U}} t_i$. □

1.20. REMARK. It is not true that the set of types over A is compact with respect to the logic topology. Indeed, for each $n > 0$, the set $[\,\|x\| \geq n\,]$ is closed in the logic topology. The family of sets of this form has the finite intersection property. However,

$$\bigcap_{n>0} [\,\|x\| \geq n\,] = \emptyset.$$

10. Quantifier-Free Types

Suppose that $\mathbf{X}$ is a Banach a space structure with universe X. If $\bar{c}$ is a finite tuple of elements of X and A is a subset of X, the *quantifier-free type of $\bar{c}$ over A* is the set of formulas

$$\{\ \varphi(\bar{x}, \bar{a})\ \mid\ \varphi \text{ is quantifier-free},\ \bar{a} \in A,\ (\,\mathbf{X}, a \mid a \in A\,) \models_{\mathcal{A}} \varphi(\bar{c}, \bar{a})\ \}.$$

1.21. PROPOSITION. *Let $\mathbf{X}$ be a Banach space structure, let A be a subset of the universe of $\mathbf{X}$, and let L be a language for the structure $(\,\mathbf{X}, a \mid a \in A\,)$. The following conditions are equivalent for a set of quantifier-free positive bounded L-formulas $t(\bar{x}) = t(x_1, \ldots, x_n)$.*

(1) *There exists a Banach space structure $\mathbf{Y} \succ_{\mathcal{A}} \mathbf{X}$ and an n-tuple $\bar{c}$ of elements of the universe of $\mathbf{Y}$ such that $t(\bar{x})$ is the quantifier-free type of $\bar{c}$ over A.*

(2) (a) *There exists $M > 0$ such that the formula*

$$\bigwedge_{1\leq i\leq n} \|x_i\| \leq M$$

is in t,

(b) *Every formula in t_+ is satisfied in $(\,\mathbf{X}, a \mid a \in A\,)$,*

(c) *For every quantifier-free L-formula $\varphi(\bar{x})$, either $\varphi \in t$, or there exists $\varphi' > \varphi$ such that $\operatorname{neg}(\varphi') \in t$.*

SKETCH OF PROOF. Analogous to the proof of Proposition 1.17 □

If $\mathbf{X}$ is a Banach space structure, A is a subset of the universe of $\mathbf{X}$, and $t(\bar{x})$ is a set of quantifier-free positive formulas satisfying the equivalent conditions of Proposition 1.21, we say that t is a *quantifier-free type over* A and $\bar{c}$ *realizes* t (or $\bar{c}$ is a *realization* of t) in $\mathbf{Y}$. If $\bar{x} = x_1, \ldots, x_n$, t is called a *quantifier-free n-type.*

The *logic topology* on quantifier-free types is the topology on the set of quantifier-free types over A where the basic open neighborhoods of a quantifier-free type t are the sets of the form $[\varphi]$, with $\varphi \in t_+$. This topology is Hausdorff.

If $t(x_1, \ldots, x_n)$ is a quantifier-free type and $(c_1, \ldots, c_n)$ is a realization of t, the *norm* of t, denoted $\|t\|$, is the number $\max_{1 \le i \le n} \|c_i\|$.

1.22. PROPOSITION. *For any $M > 0$, the set of quantifier-free types of norm less than or equal to M is compact with respect to the logic topology.*

SKETCH OF PROOF. Similar to the proof of Proposition 1.19. $\square$

11. Saturated and Homogeneous Structures

Let κ be an infinite cardinal. A Banach space structure $\mathbf{X}$ is said to be *κ-saturated* if whenever A is a subset of the universe of $\mathbf{X}$ with $\operatorname{card}(A) < \kappa$, every type over A is realized in $\mathbf{X}$.

1.23. EXERCISE. Prove that if $\mathbf{X}$ is $\aleph_1$-saturated and φ is a positive bounded formula, then

$$\mathbf{X} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathbf{X} \models \varphi[a_1, \ldots, a_n].$$

1.24. PROPOSITION. *For every Banach space structure $\mathbf{X}$ and every infinite cardinal κ there exists a Banach space structure $\mathbf{Y}$ such that $\mathbf{Y} \succ_{\mathcal{A}} \mathbf{X}$ and $\mathbf{Y}$ is κ^+-saturated.*

In order to prove Proposition 1.24, let us first introduce the following terminology.

Suppose that $(I, <)$ is a linearly ordered set. A *chain* of Banach space L-structures is a family $(\mathbf{X}_i \mid i \in I)$ of Banach space L-structures such that $\mathbf{X}_i$ is a substructure of $\mathbf{X}_j$ for $i < j$. Given a chain $(\mathbf{X}_i \mid i \in I)$ of L-structures, where

$$\mathbf{X}_i = (\ X_i,\ T_{i,j},\ c_{i,k} \ \mid\ j \in J,\ k \in K\ ),$$

one can define the *union* of the family, $\bigcup_{i \in I} \mathbf{X}_i$, naturally as follows. We set

$$\bigcup_{i \in I} \mathbf{X}_i = (\ \hat{X},\ T_j,\ c_k \ \mid\ j \in J,\ k \in K\ )$$

where the space $\hat{X}$ is the norm-completion of $\bigcup_{i\in I} X_i$, and for each $j \in J$, T_j is the unique operator on $\hat{X}$ that extends $T_{i,j}$ for every $i \in I$.

If $(I, <)$ is a linearly ordered set and $(\mathbf{X}_i \mid i \in I)$ is a chain of Banach space L-structures, we say that $(\mathbf{X}_i \mid i \in I)$ is an *approximately elementary chain* if $\mathbf{X}_i \prec_{\mathcal{A}} \mathbf{X}_j$ for $i < j$. Notice that in this case, by Proposition 1.12 we have

$$\mathbf{X}_i \prec_{\mathcal{A}} \bigcup_{i\in I} \mathbf{X}_i, \qquad \text{for every } i \in I.$$

Sketch of proof of Proposition 1.24. Fix $\mathbf{X}$ and an infinite cardinal κ. Using compactness (Theorem 1.10) inductively, we construct an approximately elementary chain of structures

$$\mathbf{X} = \mathbf{X}_0 \prec_{\mathcal{A}} \mathbf{X}_1 \prec_{\mathcal{A}} \cdots \prec_{\mathcal{A}} \mathbf{X}_i \prec_{\mathcal{A}} \cdots \qquad (i < \kappa^+) \tag{1}$$

such that for every $i < \kappa^+$,

- Every type over the universe of $\mathbf{X}_i$ is realized in $\mathbf{X}_{i+1}$,
- If i is a limit ordinal, $\mathbf{X}_i = \bigcup_{j<i} \mathbf{X}_j$.

It is easy to see that $\bigcup_{i<\kappa^+} \mathbf{X}_i$ is κ^+-saturated. □

By Proposition 1.13, every ultrapower of $\mathbf{X}$ is an approximately elementary extension of $\mathbf{X}$. Can the extension $\mathbf{Y}$ of Proposition 1.24 be chosen as an ultrapower of $\mathbf{X}$? The answer is yes. When $\kappa = \aleph_1$ this is not difficult to obtain; in fact, if $\mathcal{U}$ i is a countably incomplete ultrafilter, the $\mathcal{U}$-ultrapower of $\mathbf{X}$ is an $\aleph_1$-saturated extension of $\mathbf{X}$ (See [**HI02**]). Now if $\kappa > \aleph_1$, the answer is still positive, but the proof is much more difficult. See Theorem 1.27 below.

A Banach space structure $\mathbf{X}$ is said to be *strongly κ-homogeneous* if whenever A is a subset of the universe of $\mathbf{X}$ with $\operatorname{card}(A) < \kappa$ and $f\colon A \to X$ is such that

$$(\,\mathbf{X},\, a \mid a \in A\,) \equiv_{\mathcal{A}} (\,\mathbf{X},\, f(a) \mid a \in A\,)$$

there exists a bijection $F\colon X \to X$ extending f such that

$$(\,\mathbf{X},\, a \mid a \in X\,) \equiv_{\mathcal{A}} (\,\mathbf{X},\, F(a) \mid a \in X\,),$$

i.e., F is an *automorphism* of $\mathbf{X}$.

1.25. Remark. If $\mathbf{X}$ is strongly κ-homogeneous, $c_1, \dots, c_n, d_1, \dots, d_n$ are elements of the universe of $\mathbf{X}$, and A is a subset of the universe of $\mathbf{X}$ with $\operatorname{card}(A) < \kappa$, then the following two conditions are equivalent:

(1) $\operatorname{tp}(c_1, \dots, c_n/A) = \operatorname{tp}(c_1, \dots, c_n/A)$.
(2) There is an automorphism of $\mathbf{X}$ that maps c_i to d_i $(i = 1, \dots, n)$ and fixes A pointwise.

1.26. Proposition. *For every Banach space structure* $\mathbf{X}$ *and every infinite cardinal* κ *there exists a Banach space structure* $\mathbf{Y}$ *such that* $\mathbf{Y} \succ_{\mathcal{A}} \mathbf{X}$ *and* $\mathbf{Y}$ *is strongly* κ^+*-homogeneous.*

Sketch of proof. One constructs structures as in (1) above such that whenever $i < \kappa^+$, the structure $\mathbf{X}_{i+1}$ is $(\mathrm{card}(X_i))^+$-saturated, where X_i is the universe of $\mathbf{X}_i$, and $\mathbf{Y}$ is defined as $\bigcup_{i<\kappa^+} \mathbf{X}_i$. An argument similar to the proof of Theorem 1.15 shows that $\mathbf{Y}$ is strongly κ^+-homogeneous. $\square$

A Banach space structure $\mathbf{X}$ is called κ*-special* if there exists an approximately elementary chain $(\mathbf{X}_i \mid i < \kappa)$ such that $\mathbf{X} = \bigcup_{i<\kappa} \mathbf{X}_i$ and for every $i < \kappa$ the structure $\mathbf{X}_{i+1}$ is $(\mathrm{card}(X_i))^+$-saturated, where X_i is the universe of $\mathbf{X}_i$. The argument used to prove Theorem 1.15 shows that if the language contains no operator symbols and $\mathbf{X}$ is κ^+-special, then $\mathbf{X}$ has the following property: every $(1+\epsilon)$-isomorphism between two approximately elementary substructures of $\mathbf{X}$ whose universes have density character less than κ^+ can be extended to a $(1+\epsilon)$-automorphism of $\mathbf{X}$.

1.27. Theorem. *For every Banach space structure* $\mathbf{X}$ *and every infinite cardinal* κ *there exists an ultrapower* $\hat{\mathbf{X}}$ *of* $\mathbf{X}$ *such that* $\hat{\mathbf{X}}$ *is* κ*-saturated is strongly* κ*-homogeneous.*

Theorem 1.27 is a corollary of a much more general result that was proved in [**HI02**]. The proof involves nontrivial combinatorial ideas.

12. General Normed Space Structures

In these notes we have focused on a particular class of structures of linear functional analysis, namely, Banach spaces equipped with families of operators and distinguished elements. We have called such structures Banach space structures. In the literature on analytic model theory, however, the same term has been used to denote much more general classes of structures (see below). For these notes we have worked with a restricted notion of Banach space structure in order to achieve a balance between two goals. These simpler structures are rich enough to allow us to present important applications of their model theory to classical mathematics, but at the same time they are simple enough to make the introductory material brief and focused.

We hope that this material will serve a dual purpose; first, it will provide the theoretical framework where the applications are presented, and second, it will illustrate the methods and techniques of analytic model theory, and provide the reader with both background knowledge and motivation to pursue a more detailed study, such as is presented in [**HI02**].

In [**HI02**], Henson and the author identified a general concept of *normed space structure* for which ultraproducts can be naturally defined, and presented a study of the tight connection between ultraproducts and the model theory of these structures. Below we reproduce the definition of normed space structure from [**HI02**], and give a list of examples to indicate the wide range of possibilities encompassed by this concept.

A *normed space structure* $\mathcal{M}$ consists of the following items:

(1) A family $(M^{(s)} \mid s \in S)$ of normed spaces.
(2) A collection of functions of the form

$$F\colon M^{(s_1)} \times \cdots \times M^{(s_m)} \to M^{(s_0)},$$

each of which is uniformly continuous on every bounded subset of its domain.

The normed spaces $M^{(s)}$ are called the *sorts* of $\mathcal{M}$. If every sort of $\mathcal{M}$ is a Banach space, we say that $\mathcal{M}$ is a *Banach space structure*.

The functions of $\mathcal{M}$ that have arity 0 correspond to distinguished elements of the sorts of $\mathcal{M}$. These elements are called the *constants* of $\mathcal{M}$.

1.28. Examples.

(1) Normed spaces X over $\mathbb{R}$: The sorts are X and $\mathbb{R}$, and the functions are the vector space operations, the additive identity 0_X and the norm of X, as well as the field operations, the additive identity 0 and the absolute value function on $\mathbb{R}$.
(2) Normed spaces X over $\mathbb{C}$: These can be regarded as normed space structures in several ways. For example we may add $\mathbb{C}$ as a sort together with its field structure and absolute value, and the scalar multiplication operation as a map from $\mathbb{C} \times X$ into X, as well as the inclusion map from $\mathbb{R}$ into $\mathbb{C}$. Alternately, we may simply include a unary function from X into itself, corresponding to scalar multiplication by $\sqrt{-1}$, in addition to the usual operations that come from regarding X as a normed space over $\mathbb{R}$.
(3) Normed vector lattices $(X, \vee, \wedge)$: This is the result of expanding the normed space structure corresponding to X (see above) by adding the lattice operations $\vee$ and $\wedge$ on X and the functions max and min on $\mathbb{R}$.

(4) Normed algebras: Multiplication is included as an operation; if the algebra has a multiplicative identity, it may be included as a constant.
(5) C^*-algebras: Multiplication and the *-map are included as operations.
(6) Hilbert spaces, where the inner product is included as a distinguished function.
(7) Pairs (X, X'), where X is a Banach space, X' is the dual of X, and the pairing between X and X' is included as a function.
(8) Triples (X, X', X''), where X' and X'' are the dual and the double dual of X and the pairing between X and X', the pairing between X' and X'', and the embedding $X \to X''$ are included as functions.
(9) Operator spaces, including for each $n \geq 1$ a real-valued function of n^2 arguments mapping each $n \times n$ matrix (a_{ij}) of elements of the underlying Banach space to its operator norm.
(10) If $\mathcal{M}$ is a normed space structure and a is an element of a sort of $\mathcal{M}$, then the expansion $(\mathcal{M}, a)$ is a normed space structure.
(11) If $\mathcal{M}$ is a normed space structure, and T is a bounded linear operator between sorts of $\mathcal{M}$, then the expansion $(\mathcal{M}, T)$ is a normed space structure in which T is a distinguished function.
(12) If $\mathcal{M}$ is a normed space structure, $M^{(s)}$ is a sort of $\mathcal{M}$, and A is a given subset of $M^{(s)}$, then $\mathcal{M}$ can be expanded by adding the real-valued function $x \mapsto \operatorname{dist}(x, A)$, where x ranges over $M^{(s)}$ and dist denotes the distance function with respect to the norm on $M^{(s)}$. The same can be done with subsets of finite cartesian products of sorts.

13. The Monster Model

In what follows, X will denote a Banach space structure and we will regard X as being embedded as an approximately elementary substructure in a single κ-saturated, κ-special structure, where κ is a cardinal larger than any cardinal mentioned in the proofs.[1] Following the tradition (started by Shelah), we will refer to this structure as the "monster model", and denote it $\mathfrak{C}$. Our assumption on the monster model allows us to regard all the structures approximately elementary equivalent to X as substructures of $\mathfrak{C}$, and all the realizations of types over subsets of them as living inside $\mathfrak{C}$. We

[1]Given that we are mostly interested in separable spaces, $\kappa = (2^{\aleph_0})^+$ will typically suffice.

will also assume that the language L contains constants symbols for all the elements of $\mathfrak{C}$.

If $\varphi(x_1,\dots,x_n)$ is a positive bounded formula, we denote by $\varphi(\mathfrak{C})$ the subset of $\mathfrak{C}^n$ defined by φ.

If $\bar{a}=(a_1,\dots,a_n)$ and $\bar{b}=(b_1,\dots,b_n)$ are in $\mathfrak{C}$ and A is a subset of $\mathfrak{C}$, then $\mathrm{tp}(\bar{a}/A)=\mathrm{tp}(\bar{b}/A)$ if and only if there is an isometric automorphism f of $\mathfrak{C}$ such that $f(a_i)=b_i$ for $i=1,\dots,n$ and f fixes A pointwise.

We follow the standard practice of identifying finite lists of elements of the monster model with finite sequences. For example, if $\bar{a}=(a_1,\dots,a_n)$ we write $\bar{a}\in\mathfrak{C}$ instead of $\bar{a}\in\mathfrak{C}^n$. Addition and scalar multiplication of finite sequences is meant to be taken componentwise.

Notice that the $\aleph_1$-saturation of the monster model implies that $\models$ and $\models_{\mathcal{A}}$ are equivalent on it, *i.e.*, for every positive bounded formula $\varphi(\bar{x})$ and every $\bar{a}\in\mathfrak{C}$, we have $\mathfrak{C}\models_{\mathcal{A}}\varphi(\bar{a})$ if and only if $\mathfrak{C}\models\varphi(\bar{a})$.

The terms "structure", "formula" and "consistent" stand, respectively, for "Banach space structure", " positive bounded formula", and "satisfied in the monster model".

CHAPTER 2

Semidefinability of Types

Let α be an ordinal and let A be a set. A sequence $(\, \bar{a}_i \mid i < \alpha\,)$ is *indiscernible over* A if

$$\operatorname{tp}(\bar{a}_{i(0)}, \ldots, \bar{a}_{i(n)}/A) = \operatorname{tp}(\bar{a}_0, \ldots, \bar{a}_n/A), \qquad \text{for } i(0) < \cdots < i(n) < \alpha.$$

2.1. Definition. Suppose $A \subseteq B$ and let $t(\bar{x})$ be a type over B. We say that t *splits over* A if there exist tuples $\bar{b}, \bar{c} \in B$ with $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$ and a formula $\varphi(\bar{x}, \bar{y})$ such that $\varphi(\bar{x}, \bar{b}) \in t(\bar{x})$ and $\varphi(\bar{x}, \bar{c}) \notin t(\bar{x})$.

2.2. Proposition. *Suppose that* $(\, \bar{a}_i \mid i < \gamma\,)$ *is a sequence such that*

(i) $\operatorname{tp}(\bar{a}_\alpha/A \cup \{\, \bar{a}_i \mid i < \alpha\,\}) \subseteq \operatorname{tp}(\bar{a}_\beta/A \cup \{\, \bar{a}_i \mid i < \beta\,\})$ *for* $\alpha < \beta < \gamma$,

(ii) $\operatorname{tp}(\bar{a}_\alpha/A \cup \{\, \bar{a}_i \mid i < \alpha\,\})$ *does not split over* A *for* $\alpha < \gamma$.

Then the sequence $(\, \bar{a}_i \mid i < \gamma\,)$ *is indiscernible.*

Sketch of proof. We prove by induction on n that if

$$i(0) < \cdots < i(n-1) < \gamma,$$

then

$$\operatorname{tp}(a_{i(0)}, \ldots, \bar{a}_{i(n-1)}/A) = \operatorname{tp}(\bar{a}_0, \ldots, \bar{a}_{n-1}/A).$$

For $n = 1$, this is given by (i). Assume that the result is true for n and take $i(0) < \cdots < i(n) < \gamma$. By the induction hypothesis and the fact that

$$\operatorname{tp}(\bar{a}_{i(n)}/A \cup \{\, \bar{a}_i \mid i < i(n)\,\})$$

does not split over A, for every formula $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1})$ with parameters in A, we have

$$\varphi(\bar{a}_{i(n)}, \bar{a}_{i(0)}, \ldots, \bar{a}_{i(n-1)}) \quad \text{if and only if} \quad \varphi(\bar{a}_{i(n)}, \bar{a}_0, \ldots, \bar{a}_{n-1}),$$

and by (i)

$$\varphi(\bar{a}_{i(n)}, \bar{a}_0, \ldots, \bar{a}_{n-1}) \qquad \text{if and only if} \qquad \varphi(\bar{a}_n, \bar{a}_0, \ldots, \bar{a}_{n-1}).$$

Putting together these two equivalences, we get

$$\varphi(\bar{a}_{i(n)}, \bar{a}_{i(0)}, \ldots, \bar{a}_{i(n-1)}) \quad \text{if and only if} \quad \varphi(\bar{a}_n, \bar{a}_0, \ldots, \bar{a}_{n-1}).$$

□

2.3. Definition. Suppose $A \subseteq B$. A type t over B is called *semidefinable over A* if every approximation of every finite subset of t is realized in A.

Recall that the logic topology allows us to regard sets of types as topological spaces; see page 9. (Traditionally, in first order logic, the space of types over a set B is denoted $S(B)$ because it is a Stone space. However, we will not use this notation because, in our positive bounded context, this pace is not Stone, although it is a Tychonoff space.) The notion of semidefinability is a natural topological notion: If $A \subseteq B$, a type t over B is semidefinable over A if and only if t is in the closure (relative to the logic topology) of the set of types over B that are realized in A. We thus we have the following important observation, which will be invoked liberally:

2.4. Remark. If $A \subseteq B$, a type t over B is semidefinable over A if and only if there exists a family $(\bar{a}_i)_{i\in I}$ in A and an ultrafilter $\mathcal{U}$ on I such that

$$\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/B) = t,$$

where the limit is taken in the logic topology.

2.5. Proposition. *Suppose that $A \subseteq B$. A type t over B that is semidefinable over A does not split over A.*

Proof. Suppose that $t(\bar{x})$ splits over A. Take $\bar{b}, \bar{c} \in B$ with $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$, a formula $\varphi(\bar{x}, \bar{y})$, and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{b}) \in t(\bar{x})$ and $\operatorname{neg}(\,\varphi'(\bar{x}, \bar{c})\,) \in t(\bar{x})$. Take formulas ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. Since t is semidefinable over A, there exists $\bar{a} \in A$ such that $\mathfrak{C} \models \psi(\bar{a}, \bar{b}) \wedge \operatorname{neg}(\,\psi'(\bar{a}, \bar{c})\,)$. But this contradicts the fact that $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$. □

2.6. Proposition. *Suppose that $A \subseteq B \subseteq C$ and let $t(\bar{x})$ be a type over B that is semidefinable over A.*

(1) *t has an extension $t'(\bar{x})$ over C that is semidefinable over A; furthermore, if $(\bar{a}_i)_{i\in I}$ is a family in A and $\mathcal{U}$ is an ultrafilter on I such that $\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/B) = t$, then t' can be chosen so that $\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/C) = t'$.*
(2) *If for every $n < \omega$ every n-type over A is realized in B, then t has a unique extension $t'(\bar{x})$ over C that is semidefinable over A.*

Proof. (1): If $(\bar{a}_i)_{i\in I}$ is a family in A and $\mathcal{U}$ is an ultrafilter on I such that $\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/B) = t$, we simply define t' as $\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/C)$.

(2): Suppose that $t_1(\bar{x})$ and $t_2(\bar{x})$ are distinct extensions of t over C that are semidefinable over A. Then there exist a formula $\varphi(\bar{x}, \bar{c})$ with $\bar{c} \in C$ and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{c}) \in t_1$ and $\operatorname{neg}(\varphi'(\bar{x}, \bar{c})) \in t_2$. Take $\bar{b} \in B$ such that $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$. By Proposition 2.5, t_1 does not split over A, so $\varphi(\bar{x}, \bar{b}) \in t_1 \upharpoonright B = t$; similarly, t_2 does not split over A, so $\operatorname{neg}(\varphi'(\bar{x}, \bar{c})) \in t_2 \upharpoonright B = t$. This contradicts the fact that t does not split over A. $\square$

2.7. Remark. The proof of part (1) given above uses compactness of the space of types over C in a fundamental way. However, Proposition 2.6 holds true without the compactness assumption. (This is useful in some contexts, for example, when dealing with sets of types that are not closed, or with more general logics *e.g.*, logics of infinitary formulas.) To prove part (1) without invoking compactness, let

$$\Gamma(\bar{x}) = \left\{ \operatorname{neg}(\varphi(\bar{x}, \bar{c})) \mid \bar{c} \in C \text{ and } \{ i \in I \mid \varphi(\bar{a}_i, \bar{c}) \} \notin \mathcal{U} \right\}.$$

We claim that if $\psi(\bar{x}) \in t$, ψ' is an approximation of ψ, $\bar{c} \in C$, and

$$(*) \qquad \{ i \in I \mid \varphi(\bar{a}_i, \bar{c}) \} \notin \mathcal{U},$$

then

$$(**) \qquad \{ i \in I \mid \psi'(\bar{a}_i) \wedge \operatorname{neg}(\varphi(\bar{a}_i, \bar{c})) \} \in \mathcal{U}.$$

To prove this, notice first that the hypothesis $\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/B) = t$ gives

$$\{ i \in I \mid \psi'(\bar{a}_i) \} \in \mathcal{U}.$$

Hence, if $(**)$ were false, we would have

$$\{ i \in I \mid \psi'(\bar{a}_i) \wedge \varphi(\bar{a}_i, \bar{c}) \} \in \mathcal{U};$$

but then, since

$$\{ i \in I \mid \psi'(\bar{a}_i) \wedge \varphi(\bar{a}_i, \bar{c}) \} \subseteq \{ i \in I \mid \varphi(\bar{a}_i, \bar{c}) \},$$

we would also have $\{ i \in I \mid \varphi(\bar{a}_i, \bar{c}) \} \in \mathcal{U}$, contradicting $(*)$. This proves the claim.

By the claim, $t \cup \Gamma$ is consistent; furthermore, for every L-formula $\theta(\bar{x}, \bar{y})$ and every $\bar{c} \in C$, we have either $\theta \in t \cup \Gamma$ or $\operatorname{neg}(\theta) \in t \cup \Gamma$. Let

$$t'(\bar{x}) = \{ \theta(\bar{x}, \bar{c}) \mid \bar{c} \in C \text{ and there exists } \theta' > \theta \text{ such that } \theta'(\bar{x}, \bar{c}) \in t \cup \Gamma \}.$$

Then $t'(\bar{x})$ is a type over C such that $t'_+ \subseteq t \cup \Gamma$. The claim says that

$$\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/C) = t'.$$

CHAPTER 3

Maurey Strong Types and Convolutions

3.1. Definition. A type t will be called a *Maurey strong type for A* if there exists a set $B \supseteq A$ such that

(1) t is over B,
(2) t is semidefinable over A,
(3) For every $n < \omega$, every n-type over A is realized in B.

In this case we say that t is a *Maurey strong type for A over B.*

The importance of Maurey strong types lies in the fact that if $t(\bar{x})$ is a Maurey strong type for A over B, then, by Proposition 2.6-(2), for every $C \supseteq B$ there exists a unique extension $t'(\bar{x})$ of $t(\bar{x})$ that is a Maurey strong type for A over C.

Suppose that $A \subseteq B, B'$, the type $t(\bar{x})$ is a strong type for A over B, and $t'(\bar{x})$ is a Maurey strong type for A over B'. We claim that if $\bar{b} = b_1, \dots, b_m \in B$ and $\bar{b}' = b'_1, \dots b'_m \in B'$ are such that $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{b}'/A)$, then for every formula $\varphi(\bar{x}, y_1, \dots, y_m)$ we have $\varphi(\bar{x}, \bar{b}) \in t$ if and only if $\varphi(\bar{x}, \bar{b}') \in t'$. To see this, let $t''(\bar{x})$ be the unique Maurey strong type A over $B \cup B'$ that extends both t and t'. Since t'' does not split over A (by Proposition 2.5), for every formula $\varphi(\bar{x}, \bar{b})$, we have $\varphi(\bar{x}, \bar{b}) \in t$ iff $\varphi(\bar{x}, \bar{b}) \in t''$ iff $\varphi(\bar{x}, \bar{b}') \in t''$ iff $\varphi(\bar{x}, \bar{b}') \in t'$.

The preceding observation allows us to think of Maurey strong types as "types over the space of types of A." Given a set A, we may choose a superset B of A such that all Maurey strong types for A under consideration are over B. (Thus, B acts as a kind of monster model for Maurey strong types for A.)

3.2. Remark. By Remark 2.4 and the preceding observation, $t(\bar{x})$ is a strong type for A if and only if there exist a unique extension $\mathbf{t}(\bar{x})$ of t to the monster model, a family $(\bar{a}_i)_{i\in I}$ in A, and an ultrafilter $\mathcal{U}$ on I such that

$$\mathbf{t} = \lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i, \mathfrak{C}),$$

where the limit is taken in the logic topology.

We now define a binary operation on Maurey strong types called the *convolution* operation.

3.3. PROPOSITION. *Let $t(\bar{x})$ and $t'(\bar{x})$ be Maurey strong types for A over some $B \supseteq A$, and define a type $t * t'$ over B as follows. Let $\bar{c}$ be a realization of t, let c' be a realization of the unique extension of t' to a Maurey strong type for A over $B \cup \{\bar{c}\}$, and define*

$$t * t'(\bar{x}) = \mathrm{tp}(\bar{c} + \bar{c}'/B).$$

Then,

(1) *$t * t'$ is a Maurey strong type for A.*
(2) *The definition of $t * t'$ is independent of the particular choice of $\bar{c}$ and $\bar{c}'$.*

PROOF. Let $\mathbf{t}$ be the unique extension of t to the monster model such that $\mathbf{t}$ is semidefinable over A, and similarly let $\mathbf{t}'$ be the unique extension of t' to the monster model such that $\mathbf{t}'$ is semidefinable over A. Pick families $(\bar{a}_i)_{i\in I}$ and $(\bar{a}_j)_{j\in J}$ in A and ultrafilters $\mathcal{U}, \mathcal{V}$ such that

$$\mathbf{t} = \lim_{i,\mathcal{U}} \mathrm{tp}(a_i/\mathfrak{C}),$$

$$\mathbf{t}' = \lim_{j,\mathcal{V}} \mathrm{tp}(a_j/\mathfrak{C}).$$

Then, if $\mathbf{t} * \mathbf{t}'$ is the unique extension of $t * t'$ to the monster model such that $\mathbf{t} * \mathbf{t}'$ is semidefinable over A, we have

$$\mathbf{t} * \mathbf{t}'(\bar{x}) = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \mathrm{tp}(\bar{a}_i + \bar{a}_j'/\mathfrak{C}).$$

This shows that $t*t'$ is semidefinable over A and its definition is independent of $\bar{c}$ and $\bar{c}'$. ☐

The proof of Proposition 3.3 provides a handy recipe to compute the convolution of two Maurey strong types; namely, if t, t' are Maurey strong types for A, $\mathbf{t}$, $\mathbf{t}'$, $\mathbf{t} * \mathbf{t}'$ are, respectively, the unique extensions of t, t', $t * t'$ to the monster model that are semidefinable over A, and

$$\mathbf{t} = \lim_{i,\mathcal{U}} \mathrm{tp}(a_i/\mathfrak{C}),$$

$$\mathbf{t}' = \lim_{j,\mathcal{V}} \mathrm{tp}(\bar{a}_j/\mathfrak{C}),$$

where the families $(\bar{a}_i)_{i\in I}$ and $(\bar{a}_j)_{j\in J}$ are in A and $\mathcal{U}, \mathcal{V}$ are ultrafilters on I, J respectively, then

$$\mathbf{t} * \mathbf{t}' = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \mathrm{tp}(\bar{a}_i + \bar{a}_j'/\mathfrak{C}).$$

Furthermore, by Proposition 3.3-(1), whenever $\mathbf{t}$, $\mathbf{t}'$ $(\bar{a}_i)_{i\in I}$, $(\bar{a}_j)_{j\in J}$, $\mathcal{U}$, and $\mathcal{V}$ are as given previously, there exists an ultrafilter $\mathcal{W}$ in $I \times J$ such that

$$\mathbf{t} * \mathbf{t}' = \lim_{(i,j),\mathcal{W}} \operatorname{tp}(\bar{a}_i + \bar{a}'_j/\mathfrak{C}).$$

An immediate consequence of these observations is the following.

3.4. Corollary. *The convolution operation is associative.*

CHAPTER 4

Fundamental Sequences

A scalar multiplication can be defined naturally on types naturally, in the following way.

4.1. DEFINITION. If $t = \mathrm{tp}(\bar{a}/A)$ and r is a scalar, we denote by rt the type $\mathrm{tp}(r\bar{a}/A)$.

4.2. PROPOSITION. *If t, t' are Maurey strong types and r is a scalar, then*

$$r(t * t') = (rt) * (rt');$$

PROOF. Immediate from the definitions. $\square$

4.3. DEFINITION. Let $t(\bar{x})$ be a Maurey strong type for A over B and let $\mathbf{t}$ be the unique extension of t to the monster model such that $\mathbf{t}$ is semidefinable over A. We will say that a sequence $(\bar{a}_n)$ is a *fundamental sequence for t* if for any choice of scalars $r_0, \dots, r_n$ we have,

$$\mathrm{tp}(r_0\bar{a}_0 + \cdots + r_n\bar{a}_n) = r_0 t * \cdots * r_n t.$$

It is immediate from this definition that a fundamental sequence for t is indiscernible over B, and that its terms realize t.

Let $t(\bar{x})$ be a Maurey strong type for A over B and let $\mathbf{t}$ be the unique extension of t to the monster model that is semidefinable over A. One can produce a fundamental sequence $(\bar{a}_n)$ for t recursively by defining $\bar{a}_n$ as a realization of $\mathbf{t} \restriction B \cup \{\bar{a}_n \mid i < n\}$. Conversely, every fundamental sequence can be generated in this fashion. Thus, if $(\bar{a}_n)$ and $(\bar{a}'_n)$ are fundamental sequences for t, then there exists an automorphism of the monster model that maps $\bar{a}_n$ to $\bar{a}'_n$ and fixes B pointwise.

4.4. DEFINITION. Let t be a Maurey strong type. The set of types of the form

$$r_0 t * \cdots * r_n t,$$

where $r_0, \dots, r_n$ are scalars, will be denoted $\mathrm{span}(t, *)$.

4.5. PROPOSITION. *Let $t(\bar{x})$ be a strong type for A over B. Then there exists a type $t' \in \overline{\mathrm{span}}(t, *)$ such that $t' = -t'$.*

If k is a positive integer, we denote by $\ell_1(k)$ the vector space $\mathbb{R}^k$ regarded as a Banach space with the ℓ_1 norm (the ℓ_1-norm of a k-tuple $(r_1, \dots, r_n) \in \mathbb{R}^k$ is $\Sigma_i |r_i|$).

Sketch of proof. For every positive bounded formula $\varphi(\bar{x})$ and every rational $\epsilon \in [0, 1)$ define a formula φ_ϵ such that:

- $\varphi_0 = \varphi$,
- $\varphi < \varphi_\epsilon < \varphi_{\epsilon'}$ if $\epsilon < \epsilon'$,
- For every approximation φ' of φ there exists $\epsilon > 0$ such that $\varphi < \varphi_\epsilon < \varphi'$.

(One way to do this is as in Section 1.7.) For every positive bounded formula $\varphi(\bar{y})$ let $\mathcal{R}_\varphi$ be the real-valued function defined on the monster model by

$$\mathcal{R}_\varphi(\bar{a}) = \begin{cases} \inf\{\epsilon \in [0,1) \mid \mathfrak{C} \models \varphi_\epsilon(\bar{a})\}, & \text{if } \{\epsilon \in [0,1) \mid \mathfrak{C} \models \varphi_\epsilon(\bar{a})\} \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

By the Perturbation Lemma (Proposition 1.6), $\mathcal{R}_\varphi$ is uniformly continuous on every bounded subset of the monster model.

Fix a fundamental sequence $(\bar{a}_n)$ for t and a finite tuple $\bar{b}$ in B.

Suppose that $\Phi(\bar{x})$ is a finite set of formulas over $\bar{b}$, say,

$$\Phi(\bar{x}) = \{\, \varphi_1(\bar{x}, \bar{b}), \dots, \varphi_n(\bar{x}, \bar{b}) \,\},$$

and define a map $\mathcal{R}^\Phi : \ell_1(n+1) \to \mathbb{R}^n$ as follows: For $(r_1, \dots, r_{n+1}) \in \ell_1(n+1)$, let

$$\mathcal{R}^\Phi(r_1, \dots, r_{n+1}) =$$
$$\big(\, \mathcal{R}_{\varphi_1(\Sigma_i r_i \bar{a}_i, \bar{y})}(\bar{b}) - \mathcal{R}_{\varphi_1(-\Sigma_i r_i \bar{a}_i, \bar{y})}(\bar{b}), \dots, \mathcal{R}_{\varphi_n(\Sigma_i r_i \bar{a}_i, \bar{y})}(\bar{b}) - \mathcal{R}_{\varphi_n(-\Sigma_i r_i \bar{a}_i, \bar{y})}(\bar{b}) \,\big).$$

Notice that the map $\mathcal{R}^\Phi$ is antipodal, *i.e.*, for $(r_1, \dots, r_{n+1}) \in \ell_1(n+1)$ we have

$$\mathcal{R}^\Phi(-r_1, \dots, -r_{n+1}) = -\mathcal{R}^\Phi(r_1, \dots, r_{n+1}).$$

By the Borsuk-Ulam antipodal map theorem, there exists a point $(r_1^\Phi, \dots, r_{n+1}^\Phi)$ in the unit sphere of $\ell_1(n+1)$ such that

$$\mathcal{R}^\Phi(r_1^\Phi, \dots, r_{n+1}^\Phi) = 0.$$

Note that for $k = 1, \dots, n$,

$$\mathfrak{C} \models \varphi_k(\Sigma_i r_i^\Phi \bar{a}_i) \quad \text{if and only if} \quad \mathfrak{C} \models \varphi_k(-\Sigma_i r_i^\Phi \bar{a}_i).$$

Therefore, by compactness (Theorem 1.10), if $\mathcal{U}$ is an ultrafilter on the set of all finite subsets $\Phi(\bar{x})$, there exist a type $t'(\bar{x})$ over B and of formulas over B such that

$$\lim_{\Phi,\mathfrak{U}} \operatorname{tp}(r_1^{\Phi}\bar{a}_1, \dots, r_{\operatorname{card}(\Phi)}^{\Phi}\bar{a}_{\operatorname{card}(\Phi)} \,/\, B) = t'(\bar{x}).$$

The type t' is as desired. □

4.6. Definition. A type t is called *symmetric* if $t = -t$.

4.7. Remark. Proposition 4.5 shows that symmetric types exist. Furthermore, the proof of 4.5 shows that given any Maurey strong type t, a symmetric type can be found as a limit of types of the form $r_1 t * \cdots * r_n t$, where $\sum |r_i| = 1$.

4.8. Proposition. *If t is a symmetric Maurey strong type over B and $(\bar{a}_n)$ is a fundamental sequence for t, then*

$$\operatorname{tp}(r_0\bar{a}_0 + \cdots + r_n\bar{a}_n) = \operatorname{tp}(\pm r_0\bar{a}_0 + \cdots + \pm r_n\bar{a}_n).$$

Proof. Immediate. □

CHAPTER 5

Quantifier-Free Types Over Banach Spaces

We begin this chapter by establishing some notational conventions.

Hereafter, we shall focus our attention on quantifier-free types. Thus, hereafter, the word "type" will stand for "quantifier-free type". If $\bar{a}$ is a finite tuple and C is a subset of the monster model, $\operatorname{tp}(\bar{a}/C)$ will denote the quantifier-free type of $\bar{a}$ over C.

The type of a tuple $(a_0, \dots, a_n)$ over a set C is completely determined by the types of the elements of the linear span of $\{a_0, \dots, a_n\}$. For many purposes, this will allows us to concentrate our attention on types of elements of the monster model, rather than tuples. Thus, unless the contrary is specified, the word "type" will be used to refer to quantifier-free 1-types.

If a be an element and C is a subset of the monster model, the quantifier-free type of a over C is completely determined by the formulas of the form

$$\|a + c\| \leq M, \qquad \|a + c\| \geq M,$$

where M is a positive rational and c is an element of the linear span of C. Since the function $c \mapsto \|a + c\|$ is uniformly continuous, it has a unique extension to the closed span of C. Thus, we can assume without loss of generality that C is a Banach space.

Recall that the *norm* of a 1-type is the norm of an element realizing the type.

5.1. Remark. If $M > 0$, the set of types of norm less than or equal to M is compact; this is in fact a restatement of the compactness theorem (Theorem 1.10), but it can be proved easily, using ultraproducts, as follows. If $(\operatorname{tp}(a_i/X))_{i\in I}$ is a family of types with $\|a_i\| \leq M$ and $\mathcal{U}$ is an ultrafilter of I, then $\lim_{i,\mathcal{U}} \operatorname{tp}(a_i/X)$ is exactly the type over X realized in the $\mathcal{U}$-ultrapower of $\overline{\operatorname{span}}\{\, X \cup \{a_i \mid i \in I\}\,\}$ by the element represented by the family $(a_i)_{i\in I}$.

The quantifier-free type of a over a Banach space X can be identified with the real-valued function

$$x \mapsto \|x + a\| \qquad (x \in X).$$

Furthermore, it is easy to see that in this identification the logic topology corresponds exactly to the product topology inherited from $\mathbb{R}^X$. The preceding remark shows that the space of types over X corresponds the closure of the set of realized types (*i.e.*, the types of the form $\mathrm{tp}(a/X)$, where $a \in X$). Thus, the density character of the space of quantifier-free types over X equals the density character of X, an in particular, the space of quantifier-free types is separable if X is separable.

5.2. Proposition. *Let X be a separable Banach space and let τ be a real-valued function on X. Then the following conditions are equivalent:*

(1) *τ is the function corresponding to a quantifier-free 1-type over X.*
(2) *There exists a sequence (x_n) in X such that*

$$\tau(x) = \lim_{n\to\infty} \|x_n + x\|, \qquad \textit{for every } x \in X.$$

Proof. Notice that if (x_n) is as in (2), then (x_n) is bounded. Hence, (2) $\Rightarrow$ (1) follows from Remark 5.1. To prove (1) $\Rightarrow$ (2), suppose that τ corresponds to $\mathrm{tp}(c/X)$. Then let $\{\, d_n \mid n \in \omega \,\}$ be a dense subset of X. Since every approximation formula of every formula in $\mathrm{tp}(c/X)$ is satisfied in X, we can find a sequence (x_n) in X such that

$$|\ \|x_n + d_k\| - \|c + d_k\|\ | < \frac{1}{n+1}, \qquad \text{for } k = 0, \dots, n.$$

Then we have $\lim_{n\to\infty} \|x_n + x\| = \|c + x\| = \tau(x)$ for every $x \in X$. $\square$

5.3. Definition. Let $t(x)$ be a type over a Banach space Y. A sequence (x_n) in Y is called *approximating* for t if

$$\lim_{n\to\infty} \mathrm{tp}(x_n/Y) = t(x).$$

We also say that (x_n) *approximates* t.

5.4. Proposition. *Every bounded sequence in a separable Banach space X has a subsequence that approximates some type over X.*

Sketch of proof. By Remark 5.1 and Proposition 5.2. $\square$

5.5. Proposition. *Let X be a separable Banach space and let Y be a separable superspace of X. Then the following conditions are equivalent:*

(1) *$\mathrm{tp}(a/Y)$ is semidefinable over X.*
(2) *There exists a sequence in X that approximates $\mathrm{tp}(a/Y)$.*

PROOF. (2) $\Rightarrow$ (1) is clear. We prove (1) $\Rightarrow$ (2). Let $\{ d_n \mid n \in \omega \}$ be a dense subset of Y. Since $\operatorname{tp}(a/Y)$ is semidefinable over X, we can find a sequence (x_n) in X such that

$$| \, \|x_n + d_k\| - \|a + d_k\| \, | < \frac{1}{n+1}, \qquad \text{for } k = 0, \dots, n.$$

Clearly, $\lim_{n\to\infty} \operatorname{tp}(x_n/Y) = \operatorname{tp}(a/Y)$. $\square$

CHAPTER 6

Digression: Ramsey's Theorem for Analysis

In this chapter we discuss a form of Ramsey's Theorem that was used by A. Brunel and L. Sucheston [**BS74**] to produce 1-subsymmetric sequences (*i.e.*, quantifier-free indiscernible sequences). The method of Brunel and Sucheston has since then become standard in Banach space geometry; H. P. Rosenthal called it *the Ramsey principle for analysts*; see [**Ros86**].

6.1. Proposition. *Let* $(a_{m,n})_{m,n<\omega}$ *be an infinite matrix of real numbers such that* $\lim_n a_{m,n}$ *exists for every* m*, and*

$$\lim_m \lim_n a_{m,n} = \alpha.$$

Then there exist $k(0) < k(1) < \dots$ *such that*

$$\lim_{i<j} a_{k(i),k(j)} = \alpha.$$

Proof. By definition, for every $\epsilon > 0$ there exists a positive integer M_ϵ such that

$$m \geq M_\epsilon \qquad \text{implies} \qquad |\lim_n a_{m,n} - \alpha| \leq \epsilon.$$

Also, for every $\epsilon > 0$ and every fixed integer $\hat{m}$ there exists $N_\epsilon^{\hat{m}}$ such that

$$n \geq N_\epsilon^{\hat{m}} \qquad \text{implies} \qquad |a_{\hat{m},n} - \lim_n a_{\hat{m},n}| \leq \epsilon.$$

Take $k(0) < k(1) < \dots$ such that

$$k(0) \geq M_1,$$
$$k(l+1) \geq \max\{\, M_{2^{-l}}, N_{2^{-l}}^{k(0)}, \dots, N_{2^{-l}}^{k(l)} \,\}.$$

It is easy to see that

$$i < j \qquad \text{implies} \qquad |a_{k(i),k(j)} - \alpha| \leq 1/2^{i-1}.$$

$\square$

We need the multidimensional version of Proposition 6.1. The proof is similar. (It can also be easily derived from Proposition 6.1 by induction and diagonalization.)

6.2. PROPOSITION. *Let*

$$(\; a_{m_1,m_2,\dots,m_d} \;\mid\; (m_1,m_2,\dots,m_d) \in \omega^d \;)$$

be a family of real numbers such the iterated limits

$$\lim_{m_d} \dots \lim_{m_1} \; a_{m_1,m_2,\dots,m_d}$$

exist. Then there exist $k(0) < k(1) < \dots$ *such that*

$$\lim_{i_1 < i_2 < \dots < i_d} a_{k(i_1),k(i_2),\dots,k(i_d)} = \lim_{m_d} \dots \lim_{m_1} \; a_{m_1,m_2,\dots,m_d}.$$

CHAPTER 7

Spreading Models

Let X be a Banach space and let Y be a superspace of X such that every quantifier-free 1-type over X is realized in Y. The proof of Proposition 2.6 shows that every quantifier-free 1-type over Y that is semidefinable over X has a unique extension over the monster model that is semidefinable over X. Thus, for the quantifier-free, 1-type context (on which we are now focusing our attention), we may define Maurey strong types for X (see Chapter 3) as types that are semidefinable over X and whose domain is a superspace of X where all 1-types over X are realized.

7.1. PROPOSITION. *Suppose that (x_n) is a bounded sequence in a separable Banach space X and that no subsequence of (x_n) converges, and let Y be a subspace of X where every type over X is realized. Then there exists a Maurey strong type $t(x)$ for X over Y and a subsequence (x'_n) of (x_n) such that whenever $r_0, \ldots, r_k$ are scalars,*

$$\lim_{n_k} \ldots \lim_{n_0} \operatorname{tp}(r_0 x'_{n_0} + \cdots + r_k x'_{n_k} / X) = (r_0 t * \cdots * r_k t) \restriction X.$$

PROOF. By taking a subsequence if necessary, we can assume that (x_n) approximates a type t_0 over X. Let t be a Maurey strong type for X over Y extending t_0 over Y, let (a_n) be a fundamental sequence for t, and let $\mathbf{t}$ be the unique extension of t to the monster model such that $\mathbf{t}$ is semidefinable over X. By Proposition 5.5, there exists a subsequence (x'_n) of (x_n) such that

$$\lim_{n} \big(x'_n \,/\, X \cup \{a_n \mid n < \omega\} \big) = \mathbf{t} \restriction (X \cup \{a_n \mid n < \omega\}).$$

The sequence (x'_n) is as required. $\square$

7.2. REMARK. The conclusion of Proposition 7.1 says that whenever (a_n) is a fundamental sequence for t, $r_0, \ldots, r_k$ are scalars, and $x \in X$,

$$\lim_{n_k} \ldots \lim_{n_0} \|r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x\| = \|r_0 a_0 + \cdots + r_k a_k + x\|.$$

By Ramsey's Theorem (Proposition 6.2) we can assume that

$$\lim_{n_k<\cdots<n_0} \|r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x\| = \|r_0 a_0 + \cdots + r_k a_k + x\|.$$

In this latter case, we call the Banach space spanned by X and $\{a_n \mid n < \omega\}$ the *spreading model* approximated by the sequence (x_{n_k}) over X. The sequence (a_n) is called the *fundamental sequence* of the spreading model. Clearly, the fundamental sequence of a spreading model over X is quantifier-free indiscernible over X. See the historical remarks for further comments on the concept of spreading model.

7.3. Definition. Let (x_n) be a sequence in a Banach space. We say that (y_n) is a *sequence of blocks* of (x_n) if there exist finite subsets $F_0, F_1, \dots$ of ω such that $\max F_n < \min F_{n+1}$ and $y_n \in \operatorname{span}\{\, x_k \mid k \in F_n \,\}$ for every $n < \omega$. If (y_n) is a sequence of blocks of (x_n) we say that (y_n) is *normalized* if $\|y_n\| = 1$ for every n.

7.4. Proposition. *Suppose that (x_n) is a bounded sequence in a separable Banach space X and that no normalized sequence of blocks of (x_n) converges, and let Y be a subspace of X where every type over X is realized. Then there there exists a symmetric Maurey strong type $t(x)$ for X over Y and a subsequence (x'_n) of (x_n) such that whenever $r_0, \dots, r_k$ are scalars,*

$$\lim_{n_k} \dots \lim_{n_0} \operatorname{tp}(r_0 x'_{n_0} + \cdots + r_k x'_{n_k}/X) = (r_0 t * \cdots * r_k t) \restriction X.$$

Proof. By Propositions 4.5 and 7.1. □

Two sequences (a_n) and (b_n) are called *1-equivalent* if the map $a_n \mapsto b_n$ determines an isometry between the span of $\{\, a_n \mid n < \omega \,\}$ and the span of $\{\, b_n \mid n < \omega \,\}$.

7.5. Definition. A sequence (a_n) in a Banach space is said to be *1-unconditional* if whenever (ϵ_n) is a sequence such that $\epsilon_n = \pm 1$, the sequence $(\epsilon_n a_n)$ is 1-equivalent to (a_n).

By Proposition 4.8, every sequence that is fundamental for a symmetric Maurey strong type is indiscernible and 1-unconditional.

7.6. Proposition. *Suppose that (x_n) is a bounded sequence in a separable Banach space X and that no normalized sequence of blocks of (x_n) converges. Then (x_n) has a sequence of blocks that approximates a spreading model whose fundamental sequence is 1-unconditional.*

Proof. Immediate from Proposition 7.4 and the preceding remarks. □

CHAPTER 8

ℓ_p- and c_0-Types

8.1. DEFINITION. Let $t(x)$ be a Maurey strong type. If p is a real number satisfying $1 \le p < \infty$, we will say that t is an *ℓ_p-type* if

- t is symmetric,
- If $r, s \ge 0$, then $rt * st = (r^p + s^p)^{1/p} t$.

The type t is called a *c_0-type* if

- t is symmetric,
- If $r, s \ge 0$, then $rt * st = \max(r, s) t$.

8.2. DEFINITION. Let X be a Banach space and let p be a real number satisfying $1 \le p < \infty$. A sequence (a_n) is said to be *isometric over X to the standard unit basis of ℓ_p* if whenever $x \in X$ and $r_0, \dots, r_n$ are scalars,

$$\Big\| x + \sum_{i=0}^{n} r_i a_i \Big\| = \Big\| x + \big(\sum_{i=0}^{n} |r_i|^p \big)^{1/p} a_0 \Big\| .$$

The sequence (a_n) is said to be *isometric over X to the standard unit basis of c_0* if whenever $x \in X$ and $r_0, \dots, r_n$ are scalars,

$$\Big\| x + \sum_{i=0}^{n} r_i a_i \Big\| = \Big\| x + \big(\max_i |r_i| \big) a_0 \Big\| .$$

8.3. PROPOSITION. *Suppose that X is a Banach space, Y is a superspace of X, and $t(x)$ is a symmetric strong type for X over Y. Suppose also that (a_n) is a fundamental sequence for t. Then the following conditions are equivalent for a real number $p > 0$:*

(1) *$1 \le p < \infty$ and t is an ℓ_p-type.*
(2) *$1 \le p < \infty$ and (a_n) is isometric over Y to the standard unit basis of ℓ_p.*

(3) *For every $x \in Y$ and every natural number k,*

$$\begin{aligned}\Big\| x + \sum_{i=0}^{m-1} r_i a_i + (k+1)^{1/p} a_m + \sum_{i=m+1}^{n} r_i a_i \Big\| \\ = \Big\| x + \sum_{i=0}^{m-1} r_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} r_i a_{i+k} \Big\|.\end{aligned}$$

PROOF. (1) $\Rightarrow$ (2): We prove by induction on n that the first equality in Definition 8.2 holds. If $n \leq 1$, the equality is immediate. Assume $n \geq 1$.

Let (x_ν) be a net in X such that

$$\lim_{\nu} \operatorname{tp}(x_\nu / Y) = t.$$

Then,

$$\begin{aligned}\Big\| x + \sum_{i=0}^{n} r_i a_i \Big\| &= \lim_{\nu_n} \dots \lim_{\nu_2} \Big\| x + r_0 a_0 + r_1 a_1 + \sum_{i=2}^{n} r_i x_{\nu_i} \Big\| \\ &= \lim_{\nu_n} \dots \lim_{\nu_2} \Big\| x + (|r_0|^p + |r_1|^p)^{1/p} a_0 + \sum_{i=2}^{n} r_i x_{\nu_i} \Big\| \\ &= \Big\| x + (|r_0|^p + |r_1|^p)^{1/p} a_0 + \sum_{i=2}^{n} r_i a_i \Big\| \\ &= \lim_{\nu_0} \Big\| x + (|r_0|^p + |r_1|^p)^{1/p} x_{\nu_0} + \sum_{i=2}^{n} r_i a_i \Big\| \\ &= \lim_{\nu_0} \Big\| x + (|r_0|^p + |r_1|^p)^{1/p} x_{\nu_0} + \big(\sum_{i=2}^{n} |r_i|^p \big)^{1/p} a_n \Big\| \\ &= \Big\| x + (|r_0|^p + |r_1|^p)^{1/p} a_0 + \big(\sum_{i=2}^{n} |r_i|^p \big)^{1/p} a_n \Big\| \\ &= \Big\| x + \big(\sum_{i=0}^{n} |r_i|^p \big)^{1/p} a_0 \Big\|.\end{aligned}$$

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are immediate. We prove (3) $\Rightarrow$ (2).

Fix scalars $r_0, \dots, r_n$. Since t is symmetric, we can also assume that $r_0, \dots, r_n$ are nonnegative. Furthermore, by a density argument, we may assume without loss of generality that r_i^p is rational, for $i = 0, \dots, n$. We can therefore fix a positive integer M such that Mr_i^p is an integer, for $i = 0, \ldots, n$. By the indiscernibility of (a_n) over Y, for every $x \in Y$ we have

$$\begin{aligned}\Big\| M^{1/p}x + \sum_{i=0}^{n} (Mr_i^p)^{1/p} a_0 \Big\| &= \Big\| M^{1/p}x + \sum_{i=0}^{n} \sum_{j=0}^{Mr_i^p - 1} a_{i+j} \Big\| \\ &= \Big\| M^{1/p}x + \big(\sum_{i=0}^{n} Mr_i^p\big)^{1/p} a_i \Big\|.\end{aligned}$$

Dividing by $M^{1/p}$, we obtain the desired result. □

8.4. Proposition. *Suppose that X is a Banach space, Y is a superspace of X, and $t(x)$ is a symmetric strong type for X over Y. Suppose also that (a_n) is a fundamental sequence for t. Then the following conditions are equivalent:*

(1) *t is a c_0-type.*
(2) *(a_n) is isometric over Y to the standard unit basis of c_0 over Y.*
(3) *For every $x \in Y$ and every natural number k,*

$$\Big\| x + \sum_{i=0}^{m-1} r_i a_i + a_m + \sum_{i=m+1}^{n} r_i a_i \Big\| = \\ \Big\| x + \sum_{i=0}^{m-1} r_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} r_i a_{i+k} \Big\|.$$

Proof. Similar to the proof of Proposition 8.3 □

8.5. Remark. The equivalence (2) $\Leftrightarrow$ (3) in Propositions 8.3 and 8.4 holds for arbitrary (a_n). (The assumption that (a_n) is fundamental is not needed in the proof.)

CHAPTER 9

Extensions of Operators by Ultrapowers

In this chapter we prove a simple but powerful observation about ultrapowers of operators, namely, Proposition 9.3. This proposition will be used in Chapter 11 to transform indiscernible sequences. In this chapter, all Banach spaces mentioned are assumed to be complex.

Recall that the set of operators on a Banach space is a Banach space, with the norm of an operator T defined by $\sup_{\|x\|\leq 1}\|T(x)\|$. The identity operator is denoted I. Note that if T, W are operators on X, then $\|TW\| \leq \|T\|\,\|W\|$.

9.1. Proposition. *Let X be a Banach space.*

(1) *If T is an operator on X with $\|T\| < 1$, then $I - T$ is invertible.*
(2) *The set of invertible operators on X is open in the norm topology.*

Proof. (1): Let $W = \sum_n T^n$. It is easy to see that W is an operator on X and $(I-T)W = W(I-T) = I$.

(2): Suppose that W is an invertible operator on X. If T is any other operator, $\|I - TW^{-1}\| \leq \|W - T\|\,\|W^{-1}\|$. Thus, if $\|W - T\| < \|W^{-1}\|^{-1}$, then TW^{-1} is invertible by (1), and hence so is T. $\square$

The *spectrum* of an operator T on a complex Banach space is

$$\{\,\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\,\}.$$

It follows from Proposition 9.1 that the spectrum of an operator is a closed subset of $\mathbb{C}$.

9.2. Proposition. *Let T be an operator on a complex Banach space X and let λ be an element of the boundary of the spectrum of T. Then there exists an ultrapower $(\hat{X}, \hat{T})$ of (X, T) and $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}(e) = \lambda e$.*

Proof. By replacing T with $T - \lambda I$, we can assume that $\lambda = 0$. Note that then 0 is in the spectrum of T, since it is in the boundary and the spectrum is closed.

Suppose that the conclusion of the proposition is false. Then there exists $\delta > 0$ such that $\inf_{\|x\|=1} \|T(x)\| \geq \delta$. Also, since 0 is in the boundary of the spectrum of T, we can find complex numbers μ of arbitrarily small modulus such that $T - \mu I$ is invertible. Fix such μ with $|\mu| < \frac{\delta}{2}$. Then, by Proposition 9.1, the operator $1 + \mu(T - \mu I)^{-1}$ is invertible. But then so is

$$(T - \mu I)(1 + \mu(T - \mu I)^{-1}) = T,$$

which contradicts the fact that 0 is in the spectrum of T. $\square$

9.3. Proposition. *Let $(T_i \mid i \in I)$ be a family of operators on a complex Banach space X such that $T_i T_j = T_j T_i$ for $i, j \in I$, and suppose that $(\lambda_i \mid i \in I)$ is a family of complex numbers such that λ_i is in the boundary of the spectrum of T_i, for every $i \in I$. Then there exist*

- *an ultrapower $(\hat{X}, \hat{T}_i \mid i \in I)$ of $(X, T_i \mid i \in I)$, and*
- *an element $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}_i(e) = \lambda_i e$ for every $i \in I$.*

Proof. By compactness, it suffices to consider the case when I is finite. We prove the proposition by induction on the number of elements of I. If I is a singleton, our proposition is just Proposition 9.2. Assume, then, that $I = \{1, \dots, n\}$.

By induction hypothesis, there exists an ultrapower $(\hat{X}, \hat{T}_i \mid i \leq n)$ of $(X, T_i \mid i \leq n)$ and an element $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}_i(e) = \lambda_i e$ for $i < n$. Let

$$Y = \{\, x \in \hat{X} \;\mid\; \hat{T}_i(x) = \lambda_i x \text{ for } i < n \,\}.$$

Since $\hat{T}_n$ commutes with $\hat{T}_i$ for $i < n$, we have $\hat{T}_n(Y) \subseteq Y$. For $i \leq n$, let $U_i : Y \to Y$ be the restriction of $\hat{T}_i$ to Y, and consider the structure

$$\mathbf{Y} = (Y, U_i \mid i \leq n).$$

Proposition 9.2, provides an ultrapower $(\hat{Y}, \hat{U}_i \mid i \leq n)$ of $\mathbf{Y}$ and an element $f \in \hat{Y}$ with $\|f\| = 1$ satisfying $\hat{U}_i(f) = \lambda_i f$ for $i = 1, \dots, n$. By compactness, $(\hat{Y}, \hat{U}_i \mid i \leq n)$ can be embedded in an ultrapower of $(\hat{X}, \hat{T}_i \mid i \leq n)$, so the proposition follows. $\square$

CHAPTER 10

Where Does the Number p Come From?

Our goal in the next chapters will be to find ℓ_p-like spaces inside Banach spaces. A common question is: how does the p arise? Generally, p is given by a variation of the following elementary observation.

10.1. PROPOSITION. *Let $(\lambda_n)_{n\geq 1}$ be a sequence of real numbers such that*

(i) $1 = \lambda_1 \leq \lambda_2 \leq \dots,$

(ii) $\lambda_m \lambda_n = \lambda_{mn}$.

Then, either $\lambda_n = 1$ for every n, or there exists a number $p > 0$ such that $\lambda_n = n^{1/p}$ for every n.

PROOF. Suppose $\lambda_2 > 1$ and let $p = \frac{\log 2}{\log(\lambda_2)}$. Fix integers $m, n \geq 2$. For every integer k there exists an integer $h = h(k)$ such that $m^{h(k)} \leq n^k < m^{h(k)+1}$. By (i) and (ii), we have $\lambda_m^{h(k)} \leq \lambda_n^k \leq \lambda_m^{h(k)+1}$. Hence,

$$\left| \, k\frac{\log n}{\log m} - k\frac{\log \lambda_n}{\log \lambda_m} \, \right| \leq 1.$$

By letting $k \to \infty$, we obtain

$$\frac{\log \lambda_n}{\log n} = \frac{\log \lambda_m}{\log m}.$$

Hence, $\lambda_n = n^{1/p}$. $\square$

CHAPTER 11

Block Representability of ℓ_p in Types

If t is a strong type, $u \in \operatorname{span}(t,*)$ and $u = s_0 t * \cdots * s_m t$, we denote by $r_0 u * \cdots * r_k u$ the element of $\operatorname{span}(t,*)$ given by

$$r_0 s_0 t * \cdots * r_0 s_m t * \cdots * r_k s_0 t * \cdots * r_k s_m t.$$

11.1. THEOREM. *Let t be a nonzero symmetric Maurey strong type for X. Then there exists a sequence (e_n) with the following properties:*

(1) *(e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \le p < \infty$.*

(2) *There exists a sequence (u_l) of types in* $\operatorname{span}(t,*)$ *such that for scalars $r_0, \ldots, r_k$,*

$$\operatorname{tp}(\, r_0 e_0 + \cdots + r_k e_k \,/\, X \,) = \lim_l \, (r_0 u_l * \cdots * r_k u_l) \restriction X.$$

In the proof of Theorem 11.1, we will use Banach space operators and refer to Chapter 9. Since the spectrum of an operator is guaranteed to be nonempty only when the field of scalars is the field of complex numbers, we will use the concept of *complexification* of a Banach space, which we explain below.

Let t be a nonzero symmetric symmetric Maurey strong type for X over Y. Suppose that $(\Sigma, <)$ is an ordered set and $(a_\nu)_{\nu \in \Sigma}$ is a family such that, for scalars $r_0, \ldots, r_k$,

$$\operatorname{tp}(\, r_0 a_{\nu_0} + \cdots + r_k a_{\nu_k} \,/\, Y \,) = r_0 t * \cdots * r_k t, \qquad \text{if } \nu_0 < \cdots < \nu_k \text{ are in } \Sigma$$

(so (a_ν) is necessarily indiscernible over X). Let $Z = \overline{\operatorname{span}}\{\, a_\nu \mid \nu \in \Sigma \,\}$. Then Z can be extended to a complex Banach space naturally by defining, for $r_0, \ldots, r_k \in \mathbb{C}$,

$$\|r_0 a_{\nu_0} + \cdots + r_k a_{\nu_k}\| = \||r_0| a_{\nu_0} + \cdots + |r_k| a_{\nu_k}\|.$$

The resulting complex Banach space is called the *complexification* of Z and is denoted $Z^{\mathbb{C}}$. Since t is symmetric, the norm of $Z^{\mathbb{C}}$ extends that of Z. If $z = \sum r_i a_{\nu_i} \in Z^{\mathbb{C}}$, the element $\sum |r_i| a_{\nu_i} \in Z$ is denoted $|z|$ and called the *modulus* of z.

PROOF OF THEOREM 11.1. Let $(a_q)_{q\in\mathbb{Q}\cap(0,1)}$ be an indiscernible family such that for scalars $r_0,\dots,r_k$,

$$\operatorname{tp}(\, r_0 a_{q_0} + \cdots + r_k a_{q_k} \,/\, X\,) = r_0 t * \cdots * r_k t, \qquad \text{if } q_1 < \cdots < q_k.$$

Let $Z = \overline{\operatorname{span}}\{\, a_i \mid i \in I\,\}$. For each positive integer n define an operator $T_n \colon Z^{\mathbb{C}} \to Z^{\mathbb{C}}$ as follows. If $q_0 < \cdots < q_k$ are in $\mathbb{Q}\cap(0,1)$,

$$T_n\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big) = \sum_{j=0}^{n-1}\sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n}+\frac{j}{n}}.$$

We show that for every m, n,

(i) $T_n(z) \le n\|z\|$, for $z \in Z^{\mathbb{C}}$,
(ii) $T_m \circ T_n = T_{mn}$,
(iii) $\|T_n(z)\| \le \|T_{n+1}(z)\|$, for $z \in Z^{\mathbb{C}}$.

Properties (i) and (ii) follow from the indiscernibility of (a_q). To prove (iii), notice that since t is symmetric (and (a_q) is indiscernible), for $q_0 < \cdots < q_{k+1}$ in $\mathbb{Q}\cap(0,1)$ we have

$$\begin{aligned}
2&\Big\|T_n\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\| \\
&= \Big\|\sum_{j=0}^{n}\sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n+1}+\frac{j}{n+1}} + \sum_{j=0}^{n-1}\sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n+1}+\frac{j}{n+1}} - \sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n+1}+\frac{n}{n+1}}\Big\| \\
&\le \Big\|T_{n+1}\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\| + \Big\|T_{n+1}\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\| \\
&= 2\Big\|T_{n+1}\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\|.
\end{aligned}$$

Now we apply Proposition 9.3 to find an extension $(\,\hat{Z}, \hat{T}_n \mid n \ge 1,)$ of $(\,Z^{\mathbb{C}}, T_n \mid n \ge 1)$, a sequence (λ_n) of complex numbers, and a nonzero element $e \in \hat{Z}$ such that $\hat{T}_n(e) = \lambda_n e$. We now argue that λ_n can be taken in $\mathbb{R}$, and furthermore, positive.

By the definition of modulus in $Z^{\mathbb{C}}$ for $z \in Z^{\mathbb{C}}$ we have

$$\|T_n(|z|) - |\lambda_n||z|\| \le \|T_n(z) - \lambda_n z\|.$$

Hence, the same inequality remains true if $Z^{\mathbb{C}}$ is replaced by $\hat{Z}$ and T_n by $\hat{T}_n$. Therefore λ_n can be replaced by $|\lambda_n|$. From now on, we forget about the complexification of Z and switch our attention back to Z.

By Proposition 10.1 and (ii)–(iii), we conclude that either $\lambda_n = 1$ for every n, or there exists a real number $p > 0$ such that $\lambda_n = n^{1/p}$.

Let (z_l) be a sequence in the linear span of (a_q) such that

$$\lim_l \operatorname{tp}(z_l/X) = \operatorname{tp}(e/X), \qquad \lim_l T_n(z_l) = \lambda_n z_l,$$

and fix a type $u_l \in \operatorname{span}(t, *)$ such that $\operatorname{tp}(z_l/X) = u_l$.

Let $\{ c_n \mid n < \omega \}$ be a set of new constants and let $\Gamma(c_n)_{n<\omega}$ be a set of sentences expressing the following facts:

(iv) $\operatorname{tp}(r_0 c_0 + \cdots + r_k c_k \,/\, X) = \lim_l (r_0 u_l * \cdots * r_k u_l) \restriction X$ for any scalars $r_0, \ldots, r_k$,

(v) If $x \in X$ and $r_0, \ldots, r_n$ are scalars,

$$\left\| x + \sum_{i=0}^{m-1} r_i c_i + \lambda_{k+1} c_m + \sum_{i=m+1}^{n} r_i c_i \right\| = \left\| x + \sum_{i=0}^{m-1} r_i c_i + \sum_{i=m}^{m+k} c_i + \sum_{i=m+1}^{n} r_i c_{i+k} \right\|.$$

Every finite finite subset of $\Gamma(c_n)_{n<\omega}$ is realized in Z by interpreting the constants with z_l for sufficiently large l. Let $(e_n)_{n<\omega}$ realize $\Gamma(c_n)_{n<\omega}$. By Remark 8.5, if $\lambda_n = n^{1/p}$, then $1 \leq p < \infty$ and (e_n) is isometric over X to the standard unit basis of ℓ_p; otherwise $\lambda_n = 1$ for every n and (e_n) is isometric to c_0 over X. $\square$

CHAPTER 12

Krivine's Theorem

If $(a_0,\dots,a_k)$ and $(b_0,\dots,b_k)$ are finite sequences, X is a Banach space, and $\epsilon > 0$, we write

$$\operatorname{tp}(\,a_0,\dots,a_k \,/\, X\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,b_n,\dots,b_k \,/\, X\,)$$

and say that the types $\operatorname{tp}(\,a_0,\dots,a_k \,/\, X\,)$ and $\operatorname{tp}(\,b_0,\dots,b_k \,/\, X\,)$ are $(1+\epsilon)$-*equivalent over* X if there exists a $(1+\epsilon)$-isomorphism f from $\overline{\operatorname{span}}\{\,\{\,a_i \mid i \le k\,\} \cup X\,\}$ onto $\overline{\operatorname{span}}\{\,\{\,b_i \mid i \le k\,\} \cup X\,\}$ such that $f(a_i) = b_i$ for $i = 1,\dots,k$ and f fixes X pointwise.

12.1. Proposition. *Let (a_n) be a fundamental sequence for a nonzero symmetric Maurey strong type for a Banach space X. Then there exists a sequence (e_n) such that*

(1) *(e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \le p < \infty$.*
(2) *For every $\epsilon > 0$ and every $k \in \omega$ there exist blocks $b_0,\dots,b_k$ of (a_n) satisfying*

$$\operatorname{tp}(\,e_0,\dots,e_k \,/\, X\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,b_0,\dots,b_k \,/\, X\,).$$

Proof. Suppose (a_n) is fundamental for a symmetric Maurey strong type t for X. By Theorem 11.1 there exists a sequence (e_n) such that

(1) (e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \le p < \infty$.
(2) There exists a sequence (u_l) of types in $\operatorname{span}(t,*)$ such that for scalars $r_0,\dots,r_k$,

$$\operatorname{tp}(\,r_0e_0 + \cdots + r_ke_k \,/\, X\,) = \lim_l\,(r_0u_l * \cdots * r_ku_l) \restriction X.$$

Fix $\epsilon > 0$ and $k \in \omega$. By (2) and the fact that the unit ball of $(\mathbb{R}^k, \|\ \|_\infty)$ is compact, we find blocks $b_0,\dots,b_k$ of (a_n) such that whenever $r_0,\dots,r_k$ are scalars,

$$\operatorname{tp}(\,r_0e_0 + \cdots + r_ke_k \,/\, X\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,r_0b_0 + \cdots + r_kb_k \,/\, X\,).$$

The conclusion of the proposition now follows. □

A sequence (e_n) is *block finitely representable* in a sequence (a_n) if for every $\epsilon > 0$ and every $k < \omega$ there exist blocks $e_0, \dots, e_k$ of (a_n) such that

$$\mathrm{tp}(\, e_0, \dots, e_k \,/\, \emptyset\,) \overset{1+\epsilon}{\sim} \mathrm{tp}(\, b_0, \dots, b_k \,/\, \emptyset\,).$$

12.2. Theorem (Krivine's Theorem). *Let (x_n) be a bounded sequence in a Banach space such that no normalized sequence of blocks of (x_n) converges. Then, either there exists p with $1 \le p < \infty$ such that ℓ_p is block finitely representable in (x_n), or c_0 is block finitely represented in (x_n).*

Proof. After replacing (x_n) with a sequence of blocks of it if necessary, Proposition 7.4 allows us to fix a symmetric Maurey strong type $t(x)$ for X such that whenever $r_0, \dots, r_k$ are scalars,

$$\lim_{n_k < \cdots < n_0} \mathrm{tp}(r_0 x_{n_0} + \cdots + r_k x_{n_k} / X) = (r_0 t * \cdots * r_k t) \restriction X.$$

Let (a_n) be a fundamental sequence for t. Then, whenever $r_0, \dots, r_k$ are scalars,

$$(\dagger) \qquad \lim_{n_k < \cdots < n_0} \mathrm{tp}(\, r_0 x_{n_0} + \cdots + r_k x_{n_k} \,/\, X\,) = \mathrm{tp}(\, r_0 a_0 + \cdots + r_k a_k \,/\, X\,)$$

Fix $\epsilon > 0$ and $k < \omega$, and by Proposition 12.1, let (e_n) be such that

1. (e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \le p < \infty$.
2. There exist blocks $b_0, \dots, b_k$ of (a_n) with

$$(\ddagger) \qquad \mathrm{tp}(\, e_0, \dots, e_k \,/\, X\,) \overset{1+\epsilon}{\sim} \mathrm{tp}(\, b_0, \dots, b_k \,/\, X\,).$$

By $(\dagger)$, we find blocks $y_0, \dots, y_k$ of (x_n) such that

$$\mathrm{tp}(\, y_0, \dots, y_k \,/\, X\,) \overset{1+\epsilon}{\sim} \mathrm{tp}(\, b_0, \dots, b_k \,/\, X\,).$$

Putting this together with $(\ddagger)$, we obtain

$$\mathrm{tp}(\, y_0, \dots, y_k \,/\, X\,) \overset{(1+\epsilon)^2}{\sim} \mathrm{tp}(\, e_0, \dots, e_k \,/\, X\,),$$

and Krivine's Theorem follows since ϵ is arbitrary. □

CHAPTER 13

Stable Banach Spaces

A separable Banach space X is *stable* if whenever (x_m) and (y_n) are bounded sequences in X and $\mathcal{U}, \mathcal{V}$ are ultrafilters on $\mathbb{N}$,

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\|.$$

Let $\varphi(\bar{x}, \bar{y})$ be a positive bounded formula and let $\varphi'(\bar{x}, \bar{y})$ be an approximation of φ (see Section 1.4). We will say that the pair φ, φ' *has the order property* in the space X if there exist bounded sequences $(\bar{x}_m)$ and $(\bar{y}_n)$ in X such that

$$\begin{aligned} X &\models \varphi(\bar{x}_m, \bar{y}_n), && \text{if } m \leq n; \\ X &\models \mathrm{neg}(\,\varphi'(\bar{x}_m, \bar{y}_n)\,), && \text{if } m > n. \end{aligned}$$

13.1. Proposition. *A separable Banach space X is stable if and only if no pair of quantifier-free positive bounded formulas has the order property in X.*

Proof. Every quantifier-free positive formula $\varphi(\bar{x}, \bar{y})$ is equivalent to a conjunction of disjunctions of formulas of the form

$$\|\Lambda(\bar{x}, \bar{y})\| \leq r \qquad \text{or} \qquad \|\Lambda(\bar{x}, \bar{y})\| \geq r,$$

where r is a scalar and $\Lambda(\bar{x}, \bar{y})$ is a linear combination of $\bar{x}$ and $\bar{y}$. Hence, by the pigeonhole principle, a pair of quantifier-free formulas has the order property in X if and only if there exist bounded sequences (x_m) and (y_n) in X such that

$$\sup_{m<n} \big(\|x_m + y_n\|\big) \neq \inf_{m>n} \big(\|x_m + y_n\|\big).$$

But, by Ramsey's Theorem (Proposition 6.1), this is equivalent to saying that X is unstable. $\square$

Suppose that (x_m) and (x'_m) are bounded sequences in X and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ such that

$$\lim_{m,\mathcal{U}} \mathrm{tp}(x_m/X) = \lim_{m,\mathcal{U}} \mathrm{tp}(x'_m/X).$$

Then, if (y_m) is a bounded sequence in X and $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$,

$$\lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x'_m + y_n\|.$$

Similarly, if (y_n) and (y'_n) are bounded sequences in X and $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$ such that

$$\lim_{n,\mathcal{V}} \operatorname{tp}(y_n/X) = \lim_{n,\mathcal{V}} \operatorname{tp}(y'_n/X),$$

then, whenever (x_m) is a bounded sequence in X and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, we have

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y'_n\|.$$

Thus, if X is stable, we can define a binary operation $*$ on the space of types over X as follows. Let t, t' be types over X and let (x_m) and (y_n) be sequences in X such that $t = \lim_{m,\mathcal{U}} \operatorname{tp}(x_m/X)$ and $t' = \lim_{n,\mathcal{V}} \operatorname{tp}(y_n/X)$. We define

$$t * t' \;=\; \lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \operatorname{tp}(x_m + y_n/X).$$

The preceding remarks prove that this operation is well defined. This operation is called the *convolution* on the space of types of X. Notice that there is no conflict between this use of the word "convolution" and the general concept of convolution introduced in Chapter 3.

13.2. PROPOSITION. *The convolution on the space of types of a stable Banach space is commutative and separately continuous.*

PROOF. Immediate from the definitions. □

13.3. REMARK. A space X is stable if and only if there exists a separately continuous binary operation $*$ on the space of types over X that extends the addition of X in the sense that if $x, y \in X$,

$$\operatorname{tp}(x/X) * \operatorname{tp}(y/X) = \operatorname{tp}(x + y/X).$$

Examples of stable Banach spaces include the ℓ_p and L_p spaces. For a proof that these spaces are stable, we refer the reader to [**KM81**]. For further examples of stable spaces, see [**Gar82, Ray81a, Ray83b**].

13.4. REMARK. The space c_0 is not stable. For each $n < \omega$ let x_n be the nth vector of the standard unit basis of c_0, and let $y_n = x_0 + \cdots + x_n$. Then

$$\|x_n + y_m\| = \begin{cases} 1, & \text{if } m > n \\ 2, & \text{if } m \leq n. \end{cases}$$

Since the property of being stable is closed under subspaces, no stable space can contain c_0.

CHAPTER 14

Block Representability of ℓ_p in Types Over Stable Spaces

14.1. Definition. Let t be a symmetric type over X and let $1 \le p < \infty$. We will say that ℓ_p (or ℓ_∞) is *block represented in* $\operatorname{span}(t,*)$ if there exists a sequence (e_n) such that

(1) (e_n) is isometric over X to the standard unit basis of ℓ_p (respectively, c_0),
(2) There exists a sequence of types (u_l) in $\operatorname{span}(t,*)$ such that for scalars $r_0, \dots, r_k$,

$$\operatorname{tp}(\, r_0 e_0 + \cdots + r_k e_k \,/\, X\,) = \lim_l \,(r_0 u_l * \cdots * r_k u_l).$$

For a symmetric type t over X, we define

$$\mathfrak{p}[t] = \{\ p \in [1,\infty] \ \mid\ \ell_p \text{ is block represented in } \operatorname{span}(t,*)\ \}.$$

Theorem 11.1 says exactly that for every Banach space X and every nonzero symmetric type t over X, the set $\mathfrak{p}[t]$ is nonempty.

14.2. Proposition. *Suppose that X is stable. If t, t' are symmetric types over X such that $t \in \overline{\operatorname{span}(t',*)}$, then $\mathfrak{p}[t] \subseteq \mathfrak{p}[t']$.*

Proof. Suppose that $p \in \mathfrak{p}[t]$ and take (e_n), and (u_l) corresponding to p and $\operatorname{span}(t,*)$ as in Theorem 11.1. Since $u_l \in \operatorname{span}(t,*)$, we can write

$$u_l = s_0^l t * \cdots * s_{j(l)}^l t,$$

where $s_0^l, \dots, s_{j(l)}^l$ are scalars. Also, since $t \in \overline{\operatorname{span}(t',*)}$, there exists a sequence (w_m) in $\operatorname{span}(t',*)$ such that $t = \lim_m w_m$. Then for any scalars $r_1, \dots, r_k$ we have the following equalities; the last one follows from the separate continuity of the convolution.

$$\begin{aligned}
\operatorname{tp}(\, r_0e_0 + \dots + r_ke_k \,/\, X\,) \\
&= \lim_l \Big[r_0(s_0^l t * \dots * s_{j(l)}^l t) * \dots * r_k(s_0^l t * \dots * s_{j(l)}^l t) \Big] \\
&= \lim_l \Big[r_0\big(s_0^l \lim_m w_m * \dots * s_{j(l)}^l \lim_m w_m\big) * \dots \\
&\qquad \dots * r_k\big(s_0^l \lim_m w_m * \dots * s_{j(l)}^l \lim_m w_m\big) \Big] \\
&= \lim_l \Big[r_0 \lim_{m_0} \dots \lim_{m_{j(l)}} (s_0^l w_{m_0} * \dots * s_{j(l)}^l w_{m_{j(l)}}) * \dots \\
&\qquad \dots * r_k \lim_{m_0} \dots \lim_{m_{j(l)}} (s_0^l w_{m_0} * \dots * s_{j(l)}^l w_{m_{j(l)}}) \Big].
\end{aligned}$$

Now Ramsey's Theorem (Proposition 6.2) allows us to replace each of the iterated limits inside the square brackets by the same single limit. These limits can be taken out of the square brackets by the separate continuity of the convolution. Thus, by Ramsey's Theorem, we conclude $p \in \mathfrak{p}[t']$. □

14.3. PROPOSITION. *Suppose that X is stable. Then there exists a type t over X such that*

(1) *t is symmetric,*
(2) $\|t\| = 1$,
(3) $\mathfrak{p}[t'] = \mathfrak{p}[t]$ *for every type $t' \in \overline{[t]}$ of norm* 1.

PROOF. Suppose that the conclusion of the proposition is false. We construct, inductively, a sequence $(\, t_i\,)_{i<(2^{\aleph_0})^+}$ of types over X such that

(1) t_i is symmetric,
(2) $\|t_i\| = 1$,
(3) $t_i \in \overline{\operatorname{span}(t_j, *)}$ for $i > j$,
(4) $\mathfrak{p}[t_i] \subsetneq \mathfrak{p}[t_j]$ for $i > j$.

This is clearly impossible.

We construct t_i by induction on i. The case when i is a successor ordinal is given by assumption. Suppose that i is a limit ordinal. Fix an ultrafilter $\mathcal{U}$ on i. By compactness, there exists a type t' over X such that $\lim_{j<i,\mathcal{U}} t_j = t'$. Conditions (1)–(3) are satisfied by letting $t_i = t'$.

□

CHAPTER 15

ℓ_p-Subspaces of Stable Banach Spaces

Let $(\Sigma, \leq)$ be a partially ordered set. For an ordinal α we define the set Σ^α as follows:

· $\Sigma^0 = \Sigma$.

· If $\alpha = \beta + 1$,

$$\Sigma^{\alpha+1} = \{ \ \xi \in \Sigma^\alpha \ \mid \ \text{There exists } \eta \in \Sigma^\alpha \text{ with } \eta > \xi \ \}.$$

· If α is a limit ordinal,

$$\Sigma^\alpha = \bigcap_{\beta<\alpha} \Sigma^\beta.$$

Notice that $\Sigma^\alpha \subseteq \Sigma^\beta$. if $\alpha > \beta$. If $\Sigma \neq \emptyset$, the *rank* of Σ, denoted $\operatorname{rank}(\Sigma)$, is the smallest ordinal α such that $\Sigma^{\alpha+1} = \emptyset$. If such an ordinal does not exist, we say that Σ has *unbounded rank* and write $\operatorname{rank}(\Sigma) = \infty$.

15.1. Proposition. *Suppose that* $\operatorname{rank}(\Sigma) = \infty$. *Then there exists a sequence* (ξ_n) *in* Σ *such that* $\xi_0 < \xi_1 < \dots$.

Proof. Fix an ordinal α such that $\Sigma^\alpha = \Sigma^\beta$ for every $\beta > \alpha$. Take $\xi_0 \in \Sigma^\alpha$. Then $\xi \in \Sigma^{\alpha+1}$, so there exists $\xi_1 \in \Sigma^\alpha$ with $\xi_1 > \xi_0$. Now, $\xi_1 \in \Sigma^{\alpha+1}$, so there exists $\xi_2 \in \Sigma^\alpha$ with $\xi_2 > \xi_1$. Continuing in this fashion, we find (ξ_n) as desired. □

Let $X^{<\omega}$ denote the set of finite sequences of X. If $\xi, \eta \in X^{<\omega}$, we write $\xi < \eta$ if η extends ξ.

15.2. Proposition. *Suppose that* X *is stable. Then there exists* $p \in [1, \infty]$ *such that for every* $\epsilon > 0$, *the set*

$$\left\{ \ \xi \in X^{<\omega} \ \middle| \ \xi \textit{ is } (1+\epsilon)\textit{-equivalent to the standard unit basis of } \ell_p(n), \textit{ for some } n < \omega \ \right\}.$$

has unbounded rank.

Before proving the proposition, let us invoke it to prove the following famous result.

15.3. THEOREM (Krivine-Maurey, 1980). *For every stable Banach space X there exists a number $p \in [1, \infty)$ such that for every $\epsilon > 0$ there exists a sequence in X that is $(1+\epsilon)$-equivalent to the standard unit basis of ℓ_p.*

PROOF. By Propositions 15.1 and 15.2, there exists $p \in [1, \infty]$ such that for every $\epsilon > 0$ there exists a sequence in X that is $(1+\epsilon)$-equivalent to the standard unit basis of ℓ_p. But the stability of X rules out the case $p = \infty$ (see Remark 13.4), so the theorem follows. $\square$

PROOF OF PROPOSITION 15.2. Use Proposition 14.3 to fix a symmetric type t_0 over X of norm 1 and such that $\mathfrak{p}[t] = \mathfrak{p}[t_0]$ for every type $t \in \overline{\text{span}(t_0, *)}$ of norm 1. Fix $p \in \mathfrak{p}[t]$ and let

$$\Sigma[p, \epsilon] = \Big\{ \xi \in X^{<\omega} \;\Big|\; \xi \text{ is } (1+\epsilon)\text{-equivalent to the standard unit basis of } \ell_p(n), \text{ for some } n < \omega \Big\}.$$

For the sake of argument, assume $p < \infty$. (If $p = \infty$, the notational changes required in the argument are obvious.)

We construct for every ordinal α a type t_α over X such that

(1) $\|t_\alpha\| = 1$,
(2) t_α is symmetric,
(3) $t_\alpha \in \overline{\text{span}(t_\beta, *)}$ for every $\beta < \alpha$
(4) For every $\epsilon > 0$, every finite dimensional subspace E of X, and every element c with $\text{tp}(c/X) \in \text{span}(t_\alpha, *)$, the set

$$\Sigma[\epsilon, E, c] = \Big\{ (x_0, \ldots, x_n) \in X^{<\omega} \;\Big|\; \text{tp}\Big(\sum_{i=0}^{n} \lambda_i x_i \;/\; E \Big) \overset{1+\epsilon}{\sim} \Big(\sum_{i=0}^{n} |\lambda_i|^p \Big)^{1/p} \text{tp}(c/E) \text{ whenever } \lambda_0, \ldots, \lambda_n \text{ are scalars} \Big\}$$

has rank $\geq \alpha$.

Notice that if $(x_0, \ldots, x_n) \in \Sigma[\epsilon, E, c]$ and $c \neq 0$, then

$$\Big(\frac{x_0}{\|c\|}, \ldots, \frac{x_n}{\|c\|} \Big) \in \Sigma\Big[\epsilon, E, \frac{c}{\|c\|}\Big].$$

Hence, condition (4) ensures that $\text{rank}(\Sigma[p, \epsilon]) = \infty$. The other conditions are set to allow the inductive construction to go through.

Note that (3) implies that $p \in \mathfrak{p}[t_\alpha]$ for every ordinal α.

The type t_0 defined above satisfies (1)–(3). Condition (4) follows from the symmetry of t and the fact that every approximation of a type over X (in the sense of Section 1.7) is realized in any finite dimensional subspace of X.

Suppose that t_α has been defined, let (u_l) be a sequence of types of norm 1 in $\operatorname{span}(t_\alpha, *)$ that witnesses the fact that $p \in \mathfrak{p}[t_\alpha]$, and define $t_{\alpha+1} = \lim u_l$. Conditions (1)–(3) are clearly satisfied. We prove (4).

Note that for any scalars λ, μ, we have

$$(\dagger) \qquad |\lambda|^p t_{\alpha+1} * |\mu|^p t_{\alpha+1} = (|\lambda|^p + |\mu|^p)^{1/p} t_{\alpha+1}.$$

Fix $\epsilon > 0$ and a finite dimensional subspace E of X. Fix $\delta > 0$ such that $(1+\delta)^3 < 1+\epsilon$. By the preceding equation, the definition of $t_{\alpha+1}$, and the separate continuity of $*$, there is $u \in \operatorname{span}(t_\alpha, *)$ such that

$$(|\lambda|^p t_{\alpha+1} * |\mu|^p u) \upharpoonright E \overset{1+\delta}{\sim} ((|\lambda|^p + |\mu|^p)^{1/p} t_{\alpha+1}) \upharpoonright E.$$

Thus, if c is a realization of $t_{\alpha+1}$ and d is a realization of the restriction of u to F, where F is the closed linear span $E \cup \{c\}$, we have

$$\begin{aligned} (x_0, \dots, x_m) \in \Sigma[\delta, E, c], \quad & (y_0, \dots, y_n) \in \Sigma[\delta, F, d] \\ & \text{implies} \quad (x_0, \dots, x_m, y_1, \dots, y_n) \in \Sigma[\epsilon, E, c]. \end{aligned}$$

This proves that $\Sigma[\epsilon, E, c]$ for the particular case when $\operatorname{tp}(c/X) = t_{\alpha+1}$. The general case when $\operatorname{tp}(c/X) \in \operatorname{span}(t_{\alpha+1}, *)$ now follows from (†) and the fact that $\mathrm{t}_{\alpha+1}$ is symmetric.

If α is a limit ordinal, we take an ultrafilter $\mathcal{U}$ on α and define t_α as $\lim_{\beta<\alpha,\mathcal{U}} t_\beta$. □

Historical Remarks

Chapter 1

Ultraproducts were introduced by Łoś [**Łoś55**] for general first-order structures. Special cases had been used earlier by Skolem [**Sko34**] (to prove the impossibility of a finite or countably infinite first-order axiomatization of Peano Arithmetic) and Hewitt [**Hew48**] (in the context of rings of continuous real-valued functions). Keisler introduced the concepts of κ-saturated and κ-homogeneous ultrapower for an infinite cardinal κ and, assuming the generalized continuum hypothesis (GCH), proved that two structures have isomorphic ultrapowers if and only if they satisfy the same first-order sentences [**Kei61, Kei64**]. Shelah, in remarkable work, building on ideas of Kunen [**Kun72**], eliminated the GCH assumption. The Keisler-Shelah isomorphism theorem fits within Tarski's program of characterizing metamathematical concepts in "purely mathematical" terms. For a general survey of the ultraproduct construction and its role in model theory, the reader is referred to [**Kei10**].

Banach space ultraproducts were formally introduced by Dacunha-Castelle and Krivine [**DCK70, DCK72**], although Krivine had already used them prominently in his *Thèse d'Etat* [**Kri67**]), inspired by the classical ultraproduct construction from model theory. Banach space ultrapowers can be seen as a particular case of the *nonstandard hull* construction introduced by Luxemburg [**Lux69**].

In the 1970's ultrapowers became a standard tool in Banach space theory, and during the following two decades, ideas from model theory and nonstandard analysis permeated the realm of functional analysis and probability through the use of ultrapowers. The most prominent results proved by using tools that originated in model theory are Krivine's Theorem [**Kri76**] and the Krivine-Maurey result on stable Banach spaces [**KM81**]. Other noteworthy results were obtained during this period by Dacunha-Castelle [**DC72d, DC72a, DC72b, DC72c, DC75a, DC75b, DC75c**], Dacunha-Castelle and Krivine [**DCK70, DCK72, DCK75a, DCK75b**], Heinrich [**Hei78,**

Hei80a, Hei80b, Hei81, Hei84], Henson [**Hen74, HM74b, Hen75, Hen76**], Kalton [**Kal84**], Moore [**Moo76**], Heinrich and Henson [**Hei84**], Henson-Moore [**HM74c, HM74a, HM83a, HM83b**], Heinrich-Henson-Moore [**HHM83, HHM86, HHM87**], Raynaud [**Ray81a, Ray81b, Ray81c, Ray83a, Ray83b**], Levy-Raynaud [**LR84a, LR84b**], Haydon-Levy-Raynaud [**HLR85, HLR91**], Kürsten [**Kju78, Kür83, Kür84**], Schreiber [**Sch72**], and Stern [**Ste74, Ste75a, Ste75b, Ste76a, Ste76b, Ste77, Ste78**], among others.

The logical formalism of positive bounded formulas and approximate satisfaction was initially introduced by Henson [**Hen76**] in order to address the question of under what conditions two given spaces have isometric nonstandard hulls. Henson proved that two Banach spaces X and Y have isometric nonstandard hulls if and only if every positive sentence that is true in X is approximately true in Y. The model theory of Henson's logic was developed further by the author in [**Iov96, Iov97, Iov99a, Iov99b**]. In [**Iov01**] the author proved that Henson's approach provides a maximal model theory for Banach spaces, and for metric spaces in general. The first self-contained introduction to the model theory of Banach space structures via Henson's logic was presented in [**HI02**].

An equivalent way to see Banach spaces (and metric spaces in general) from the perspective of model theory without the use of approximate truth is through real-valued logic. A logic of sentences with truth values in the closed unit interval $[0, 1]$ was first proposed by Łukasiewicz and Tarski in the late 1920's. In the late 1950's, Chang [**Cha58, Cha59**] introduced the concept of MV-algebra, and used it to give an algebraic proof of the completeness of Łukasiewicz logic. MV-algebras are analogs of boolean algebras for multivalued logics; the paradigm example of such an algebra is the closed unit interval $[0, 1]$, equipped with some natural continuous operations. Chang [**Cha61**] showed how the ultraproduct construction carries over from ordinary two-valued logic to $[0, 1]$-valued logic, and generalized to the real-valued context fundamental results of Keisler [**Kei61**] on ultraproducts. Later, Chang and Keisler observed that $[0, 1]$ can be replaced with any reasonably well-behaved compact Hausdorff uniform space K; the only requirements on the logic are that the connectives and quantifiers should be uniformly continuous (an n-ary connective in this context is a function $c : K^n \to K$, and a quantifier is a function $q : \mathcal{P}(K) \to K$, where $\mathcal{P}(K)$ is endowed with the Vietoris topology). These ideas were expanded by Chang and Keisler in their extensive monograph "Continuous Model Theory" [**CK66**].

After the publication of their monograph, Chang and Kesiler did not continue their development of continuous model theory; however, four decades later, Ben Yaacov and Usvyatsov [**BYU10**] showed that the continuous $[0,1]$-valued logic used by Chang and Keisler as guide example provides an elegant reformulation of Henson's logic, not only for normed structures but also for bounded metric structures. Ben Yaacov and Usvyatsov called this formalism "continuous first-order logic." A self-contained introduction to continuous first-order logic, authored by Ben Yaacov, Berenstein, Henson, and Usvyatsov [**BYBHU08**], appeared in print before the Ben Yaacov-Usvyatsov paper did.

The basic definitions given by Ben Yaacov and Usvyatsov [**BYU10**] for for first-order continuous model theory (*e.g*, those of connective, quantifier, truth, elementary equivalence, etc.), as well as fundamental facts about approximability of connectives, come from the Chang-Keisler monograph (for the case when the set of truth values is the set $[0,1]$), and hence can be traced back to Chang's work on Łukasiewicz logic and linear MV-algebras (see [**Cha59, Cha61**]). There is, however, a subtle but important difference between both approaches: Chang and Keisler consider all structures with predicates with values in a compact truth-value space (in this case, $[0,1]$), including a distinguished predicate for the equality relation, whereas Ben Yaacov and Usvyatsov consider only continuous metric structures as defined earlier by Henson, in which a distinguished metric takes the role of the equality relation, and all the other predicates are required to be uniformly continuous with respect to this metric.

Metric structures had been used previously for the semantics of Łukasiewicz logic, but with emphasis in 1-Lipschitz continuity rather than general uniform continuity; see, for example, [**Háj98**]. It was observed in [**CI14**] that, for continuous metric structures, the logical formalism of [**BYU10**] is equivalent to the logic known as rational Pavelka logic; this is the logic that results from expanding Łukasiewicz logic with a constant connective for each rational in the closed unit interval. (See [**Háj98**].) Rational Pavelka logic is a conservative extension of the classical Łukasiewicz logic (this means, roughly, that both logics have the same expressive power); see [**HPS00**].

Related, but less general logical formalisms to study Banach spaces through real-valued sentences were proposed in the 1970's by Krivine [**Kri72, Kri74**] and Stern [**Ste76a**].

The notion of $(1+\epsilon)$-approximation and Theorem 1.15 as presented in this chapter were introduced by Heinrich and Henson in order to characterize $(1+\epsilon)$-isomorphism of Banach space ultrapowers; see [**HH86**].

Chapter 2

The notions of splitting and semidefinability in model theory are due to Shelah, and the results in this chapter are straightforward adaptations of results in [**She78, She90**].

The concepts of indiscernible sequence, saturated structure, and type have occupied a central space in model theory since its early developments, in the late 1950's. Indiscernible sequences were defined by Ehrenfeucht and Mostowski in the study of the groups of automorphisms of the models of a theory [**EM56**]. Such sequences played a crucial role in Morley's thesis [**Mor65**]. The methods introduced by Morley brought model theory into a new era and led Shelah to develop the theory of model-theoretic stability, and his formidable theory of classification of the models of a complete theory. The obligatory reference for this is Shelah's famously difficult book [**She78, She90**], although much of Shelah's output has been devoted to his classification program.

In the 1960's iterated ultrapowers became a particularly fruitful method of generating indiscernibles in set theory, thanks to groundbreaking work of Kunen, generalizing earlier work by Geifman (see [**Kun72**]).

The model-theoretic concepts of type and indiscernible sequence were used for the first time in Banach space theory by Krivine [**Kri76**]. Krivine constructed indiscernible sequences by using iterated Banach space ultrapowers (which he had defined earlier — see the Remarks on Chapter 1). He called such sequences *suites écartables*. Krivine's method of iterated Banach space ultrapowers gave an alternative construction of the concept of *spreading model*, which had been introduced a couple of years earlier by Brunel and Sucheston [**Bru74, BS74**]. Brunel sand Sucheston used Ramsey's theorem and compactness in a way reminiscent of how Ehrenfeucht and Mostowski used Ramsey's Theorem and compactness in the paper [**EM56**] where indiscernible sequences were originally introduced. (Ehrenfeucht and Mostowski used the compactness of the set $\{0,1\}$ of truth values of classical logic, whereas Brunel and Sucheston used the compactness of the interval $[0,1]$. See the remarks on Chapter 7.)

Krivine's indiscernible sequences played a fundamental role in the proof of Krivine's Theorem on the finite representability of ℓ_p in all Banah lattices [**Kri76**], and later, in the Krivine-Maurey work on stable Banach spaces [**KM81**]. See the remarks on Chapter 5.

Chapters 3 and 4

Maurey [**Mau83**] extended to unstable contexts some of the ideas introduced by Krivine and Maurey for stable Banach spaces [**KM81**], and introduced a notion of "strong type" that can be seen as a quantifier-free version of the concept of strong type defined in Chapter 3. Maurey used strong types to give a characterization of the Banach spaces that contain ℓ_1 (a similar characterization is given for c_0).

Our proof of existence of symmetric Maurey strong types via types using the Borsuk-Ulam theorem is an elaboration of an idea used by Rosenthal [**Ros83**]. See also [**Ros86**] and [**Mau83**].

The term "fundamental sequence" is borrowed from the theory of spreading models as presented by Beauzamy and Lapresté [**BL84**]. See the remarks on Chapter 7.

Chapter 5

Krivine's original definition of type for Banach spaces [**Kri76**] is as follows. If X is a Banach space and $(x_1, \dots, x_n)$ is an n-tuple of elements of X, the *type* of $(x_1, \dots, x_n)$ is the function that assigns to each n-tuple of scalars $(\lambda_1, \dots, \lambda_n)$ the norm $\|\sum_{i=1}^n \lambda_i x_i\|$. An n-type in this sense is exactly a quantifier-free n-type in continuous first-order logic (see the remarks on Chapter 1). Clearly, every type is determined by its restriction to $B_{\ell_\infty^n} = \{\, (\lambda_1, \dots, \lambda_n) \mid \sup_i |\lambda_i| \leq 1 \,\}$. Thus, the set of n-types can be regarded as a topological space, with the topology inherited from $\mathcal{C}(B_{\ell_\infty^n})$. This topology corresponds to the relativization to quantifier-free types of the topology given by the metric d on types introduced by Henson, and defined in the following way: the distance $d(p, q)$ between two n-types p and q (which are not necessarily quantifier-free) is the infimum of all the distances $\|\bar{c} - \bar{d}\|_\infty$ where $\bar{c}$ and $\bar{d}$ are n-tuples that realize p and q, respectively, in a sufficiently saturated extension of X. See [**HI02**, Section 14].

The notion of type defined by Krivine in [**Kri76**] and outlined above can be extended naturally in the following way. If X is a Banach space and $(x_1, \dots, x_n)$ is an n-tuple of elements of a superspace of X, the *type of* $(x_1, \dots, x_n)$ *over* X is the function that assigns to each $n + 1$-tuple $(x, \lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are scalars and $x \in X$, the norm

$$\left\| x + \sum_{i=1}^{n} \lambda_i x_i \right\|.$$

In their paper on stable Banach spaces [**KM81**], Krivine and Maurey reintroduced the notion of space of types for Banach space, but in this paper the definitions are slightly different because a separable Banach space X is fixed, and what is called a "type" is the type over X of an element in some ultrapower of X. Krivine and Maurey gave this definition without reference to ultrapowers in the following way: If $a \in X$, the function $\tau_a \colon X \to \mathbb{R}$ is defined by $\tau_a(x) = \|a + x\|$; the *space of types* is defined as the closure of the set $\{ \tau_a \mid a \in X \}$ in the product space $\mathbb{R}^X$. The set of types thus defined is regarded as a topological space, with the topology inherited from $\mathbb{R}^X$. This topology corresponds to the to the relativization to quantifier-free types of the topology called the "logic topology" in [**HI02**, Section 14]. The logic topology is the counterpart in continuous first-order logic of the Stone topology from classical first-order logic.

Krivine and Maurey [**KM81**] used types to produce indiscernible sequences much in the same manner that Krivine had [**Kri76**] (see the remarks on Chapter 2), but without direct use of ultrapowers and, more importantly, under the assumption that the base space X is *stable.* In model theory, stability ensures that every indiscernible sequence is *totally indiscernible*; this means that any two finite sequences of its domain of the same length have the same type. The concept of Banach space stability introduced by Krivine and Maurey is not a literal translation of model-theoretic stability, but it is exactly the condition needed to ensure that Krivine's indiscernible sequences [**Kri76**] are totally indiscernible. Total indiscernibility combined with the main ideas of [**Kri76**] allowed Krivine and Maurey to obtain the main theorem of [**KM81**] (Theorem 15.3 here).

Our definition of "approximating sequence" was borrowed from Garling's exposition of Banach space stability [**Gar82**]. (Garling does not use the clause "over X", since in [**Gar82**] the space X is regarded as fixed.)

For applications of Krivine-Maurey concept of type, see for example, [**Cha91**], [**Far88**], [**Gue86**], [**HM86**], [**Mau83**], [**Ode83**], [**Ray83b, Ray84, Ray89**], [**Ros84, Ros86**].

Chapter 6

Ramsey Theory has had a significant impact in Banach space theory. For a survey of the early applications (prior to 1980), see [**Ode80**]. For more recent surveys, which necessarily will focus on the profound contributions of Gowers, see Gowers's article in the Handbook of the Geometry of Banach Spaces [**Gow03**] and Part B of the book by Argyros and Todorčević [**AT05**].

Chapter 7

Spreading models were introduced by Brunel and Sucheston [**Bru74, BS74**] in the study of summability of sequences in Banach spaces. Brunel and Sucheston proved that whenever (x_n) is a bounded sequence in a Banach space X there exists a subsequence (x'_n) of (x_n) such that the limit

$$\lim_{n_0<\cdots<n_k} \|r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x\|$$

exists for every every $k \in \mathbb{N}$, every k-tuple of scalars $r_0, \ldots, r_k \in \mathbb{R}$, and every $x \in X$. The sequence (x'_n) is called a *good subsequence* of (x_n). We outline the argument of Brunel and Sucheston. A good subsequence (x'_n) induces a seminorm on $\mathbb{R}^\omega$ (or $\mathbb{C}^\omega$ if the space X is complex) as follows. If (e_n) is the standard basis of unit vectors in $\mathbb{R}^\omega$,

$$\|\sum_i r_i e_i\| = \lim_{n_0<\cdots<n_k} \|r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x\|.$$

This seminorm is a norm if (and only if) the sequence (x'_n) is nonconvergent. The resulting Banach space is called the *spreading model defined by the sequence* (x'_n).

Analysts use the term 1-*subsymmetric* to express the fact a sequence in a Banach space is indiscernible (with respect to quantifier-free formulas).

Chapter 8

The concepts of ℓ_p- and c_0-type were introduced by Krivine and Maurey in the context of (quantifier-free) stable Banach spaces [**KM81**].

Chapter 9

The simplification of the proof of Krivine's Theorem through the use of eigenvectors of operators (Proposition 9.2) is due to Lemberg [**Lem81**]. See the comments on Chapters 11 and 12 for further remarks on Lemberg's proof.

Chapter 11

Our proof of Theorem 11.1 is based on Lemberg's proof of Krivine's Theorem [**Lem81**]. We highlight the fact that, from a model-theoretic perspective, the main idea is quite natural.

It is natural to ask whether every Banach space has a spreading model containing ℓ_p ($1 \leq p < \infty$) or c_0. The question was answered negatively by Odell and Schlumprecht [**OS95**].

Chapter 12

The statement of Krivine's Theorem in Krivine's paper [**Kri76**] is as follows: For every bounded sequence (x_n) in a Banach lattice, either there exists p with $1 \leq p < \infty$ such that ℓ_p is block finitely representable in (x_n), or there exists a permutation of (x_n) such that c_0 is block finitely representable in (x_n). Rosenthal [**Ros78**] expounded Krivine's Theorem and obtained new results on the set of p's given by the theorem. Lemberg [**Lem81**] simplified Krivine's argument by invoking Proposition 9.2, and eliminated the need for permutations in the c_0 case.

Chapter 14

Proposition 14.3 is due to Bu [**Bu89**], and it plays a role analogous to that played by minimal cones in the Krivine-Maurey proof that every stable Banach space contains some ℓ_p almost isometrically [**KM81**].

Chapter 15

The question of which Banach spaces contain isomorphic copies ℓ_p or c_0 has played a central role in the history of Banach space geometry. The first example of a Banach space not containing ℓ_p or c_0 was constructed by Tsirelson [**Tsi74**]. On his academic web site, Tsirelson explains how his proof was inspired by the concept of forcing from set theory. Tsirelson's construction was geometric. Figiel and Johnson [**FJ74**] gave an analytic construction of the dual of this space. The method used by Figuiel and Johnson is now a standard method to construct Banach spaces with prescribed properties.

In 1981, using probabilistic methods, Aldous proved [**Ald81**] that every subspace of L_1 contains ℓ_p for some p $(1 \leq p \leq 2)$ almost isometrically. Almost immediately, Krivine and Maurey generalized the methods of Aldous to a wider class of spaces: the class of stable Banach spaces. The main theorem of their paper [**KM81**] is Theorem 15.3. The role played by types in the Krivine-Maurey proof is analogous to that played by random measures in Aldous' argument.

In their paper, Krivine and Maurey exhibit a wealth of examples of stable Banach spaces; furthermore, they prove that if X is stable, then the space $L_p(X)$ is stable, for $1 \leq p < \infty$. Garling [**Gar82**] and Raynaud [**Ray81a**] exhibited further examples.

The general theory of model-theoretic stability for Banach space structures (*e.g.*, forking, stability spectrum, etc.) was developed in [**Iov94**]. See [**Iov99a, Iov99b, Iov96, Iov97**]. Ben-Yaacov and Usvyatsov [**BYU10**] introduced the more general concept of "local stability" (*i.e.*, stability of formulas) for bounded metric spaces.

Our proof of Theorem 15.3 is based on a proof by Bu [**Bu89**]. Bu invokes the Kunen-Martin theorem from descriptive set theory (see [**Mos09**], for example), and obtains Theorem 15.3 by showing that there are types of arbitrarily high countable rank. Our argument shows that one need not invoke the Kunen-Martin Theorem if one considers values on all ordinals, rather than just countable ones.

For a survey of important of ordinal ranks in Banach space theory, see [**Ode04**].

It was observed by Krivine and Maurey that if X is a stable Banach space, then the space of types over X is *strongly separable*, *i.e.*, separable with respect to the topology of uniform convergence on bounded subsets of X (see the notes on Chapter 5 above). Odell (see [**Ode83**] or [**Ray84**]) proved that the Tsirelson space of [**FJ74**] has a strongly separable space of types; hence strong separability of the space of types does not imply stability. Later, Haydon and Maurey [**HM86**] proved that every space with a strongly separable space of types contains either a reflexive subspace or a copy of ℓ_1.

Chaatit [**Cha96**] showed that a Banach space is stable if and only if it can be embedded in the group of isometries of a reflexive Banach space.

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