ON THE MAXIMALITY OF LOGICS WITH APPROXIMATIONS

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0. Introduction

In this paper we analyze some aspects of the question of using methods from model theory to study structures of functional analysis.

By a well known result of P. Lindström, one cannot extend the expressive power of first order logic and yet preserve its most outstanding model theoretic characteristics (*e.g.*, compactness and the Löwenheim-Skolem theorem). However, one may consider extending the scope of first order in a different sense, specifically, by expanding the class of structures that are regarded as models (*e.g.*, including Banach algebras or other structures of functional analysis), and ask whether the resulting extensions of first order model theory preserve some of its desirable characteristics.

A formal framework for the study of structures based on Banach spaces from the perspective of model theory was first introduced by C. W. Henson in [8] and [6]. Notions of syntax and semantics for these structures were defined, and it was shown that using them one obtains a model theoretic apparatus that satisfies many of the fundamental properties of first order model theory. For instance, one has compactness, Löwenheim-Skolem, and omitting types theorems. Further aspects of the theory, namely, the fundamentals of stability and forking, were first introduced in [9] and [10].

The classes of mathematical structures formally encompassed by this framework are normed linear spaces, possibly expanded with additional structure, *e.g.*, operations, real-valued relations, and constants. This notion subsumes wide classes of structures from functional analysis. However, the restriction that the universe of a structure be a normed space is not necessary. (This restriction has a historical, rather than technical origin; specifically, the development of the theory was originally motivated by questions in Banach space geometry.) Analogous techniques can be applied if the universe is a metric space. Now, when the underlying metric topology is discrete, the resulting model theory coincides with first order model theory, so this logic extends first order in the sense described above. Furthermore, without any cost in the mathematical complexity, one can also work in multi-sorted contexts, so, for instance, one sort could be an operator algebra while another is, say, a metric space.

For the rest of this introduction, let us refer to the framework introduced in [8] as $\mathcal{H}$ and to the structures encompassed by $\mathcal{H}$ as *analytic structures*.

The aim of this paper is to show that if one is to develop a smooth model theory for analytic structures, then the expressive power of the logic $\mathcal{H}$ cannot be increased. (This is made more precise below.) We will not assume previous familiarity with $\mathcal{H}$, as the properties of it that will be used are very straightforward and simple to state.

The syntax of $\mathcal{H}$ is given by a class of first order formulas called *positive bounded formulas*, and the semantics by a relation between analytic structures and positive bounded formulas called *approximate satisfaction*, and denoted $\models_{\mathcal{A}}$. The role played by positive bounded formulas in $\mathcal{H}$ is analogous to that played by first order formulas in first order model theory, and the role played by the relation $\models_{\mathcal{A}}$ mirrors that played by the $\models$ in ordinary model theory. For analytic structures in general, positive bounded formulas have less expressive power than first order formulas and approximate satisfaction is weaker than ordinary satisfaction. However when the structures under consideration are discrete (*i.e.*, when the universe is a discrete metric space), $\models_{\mathcal{A}}$ coincides with $\models$ and $\mathcal{H}$ coincides with first order logic.

The notion of approximate satisfaction $\models_{\mathcal{A}}$ of $\mathcal{H}$ yields naturally relations of *approximately elementary equivalence* $\equiv_{\mathcal{A}}$ and *approximately elementary substructure* $\prec_{\mathcal{A}}$ between analytic structures (one just replaces $\models$ with $\models_{\mathcal{A}}$ in the usual definitions).

Two characteristics of $\mathcal{H}$ play a fundamental role in the development of the theory:

1. **The compactness theorem**. If Σ is a set of positive bounded sentences such that every finite subset of Σ is approximately satisfied by an analytic structure, then there exists an analytic structure that approximately satisfies every sentence of Σ.
2. **The elementary chain property**. If $\mathcal{M}_0, \mathcal{M}_1 \dots$ are analytic structures and
$$\mathcal{M}_0 \prec_{\mathcal{A}} \mathcal{M}_1 \prec_{\mathcal{A}} \dots \prec_{\mathcal{A}} \mathcal{M}_n \prec_{\mathcal{A}} \dots \qquad (n < \omega),$$
Then the structure $\mathcal{M} = \bigcup_n \mathcal{M}$ is an approximately elementary extension of $\mathcal{M}_n$ for every n.

If the context demands that the structures be complete, (*e.g.*, if the structures under consideration are Banach spaces), then in (2) one takes $\mathcal{M}$ to be the completion of $\bigcup_n \mathcal{M}$, rather than $\bigcup_n \mathcal{M}_n$ itself. (The nature of approximate satisfaction is such that an analytic structure is always an approximately elementary substructure of its completion.)

We prove the following theorem.

Theorem. *There is no logic for analytic structures that extends $\mathcal{H}$ properly and satisfies both the compactness theorem and the elementary chain property.*

We prove a rather strong version of the theorem which we now explain. The notion of approximate satisfaction of $\mathcal{H}$ is given by a topology of approximations on the class of positive bounded formulas. We show that if $\mathcal{L}$ is a logic that extends $\mathcal{H}$ properly and $\mathcal{L}$ has a notion of approximate satisfaction $\overset{\mathcal{L}}{\models}_{\mathcal{A}}$ given by a topology of approximations finer than that of $\mathcal{H}$, then $\mathcal{L}$ cannot satisfy both (1) and (2). (We actually only need a weak version of (2).)

Now, the notion of topology of approximations on the formulas of a logic includes the case when the only approximation of each formula is itself. Therefore the theorem also covers extensions $\mathcal{L}$ of $\mathcal{H}$ with no approximations. The information given by the theorem for the particular case when $\mathcal{L}$ is first order logic is not very surprising in light of the fact that the expressive power of first order logic on Banach spaces is known to be quite high [14]. Nevertheless, it seems rather striking to us that there is no logic strictly between $\mathcal{H}$ and first order satisfying both (1) and (2).

The paper contains three sections. In Section 1 we introduce the concept of logic with approximations, in Section 2 we recall some basic properties of the logic $\mathcal{H}$, and Section 3 is devoted to the proof of the main theorem.

The inspiration for the the main result was a characterization of first order logic due to P. Lindström [13].

A related but different model theoretic approach for structures based on metric spaces is the Fajardo-Keisler *neometric forcing* [11, 3, 4, 5]. In the Fajardo-Keisler approach, too, the structures under consideration are structures based on metric spaces. The emphasis, however, is on proving existence theorems (especially in stochastic analysis), rather than obtaining a first-order-like apparatus. It is nevertheless noteworthy that the neometric framework also involves approximations, and in fact the ideas in the "Simple Forcing" section of [5] are inspired by $\mathcal{H}$, as well as R. M. Anderson's *"Almost" Implies "Near"* [1].

A word about notation. Structures will be denoted by the calligraphic letters $\mathcal{M}$, $\mathcal{N}$, etc. and their universes by the corresponding Roman letters M, N, etc.

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1. Logics with Approximations

1.1. **Abstract Logics.** We assume that the reader is familiar with the concepts of *multi-sorted language* and *multi-sorted structure*. If L and L' are multi-sorted languages, a *renaming* is a bijection $r\colon L \to L'$ that maps sort symbols onto sort symbols, relation symbols onto relation symbols, and function symbols onto function symbols, and respects sorts and arities. If $r\colon L \to L'$ is a renaming and $\mathcal{M}$ is an L-structure, $\mathcal{M}^r$ denotes the structure that results from converting $\mathcal{M}$ into an L'-structure through r. We call the map $\mathcal{M} \mapsto \mathcal{M}^r$, too, a renaming.

1.1. Definition. A logic $\mathcal{L}$ consists of the following items.

(1) A class of structures, called the *structures of* $\mathcal{L}$, that is closed under isomorphisms, renamings, expansion by constants, and reducts.
(2) For each multi-sorted language L, a set $\mathcal{L}[L]$ called the *L-sentences of* $\mathcal{L}$, such that $\mathcal{L}[L] \subseteq \mathcal{L}[L']$ when $L \subseteq L'$.
(3) A binary relation $\overset{\mathcal{L}}{\models}$ between structures and sentences of $\mathcal{L}$ such that:
   (a) If $\mathcal{M}$ is an L-structure of $\mathcal{L}$ and $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi$, then $\varphi \in \mathcal{L}[L]$.
   (b) *Isomorphism Property.* If $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi$ and $\mathcal{M}$ is isomorphic to $\mathcal{N}$, then $\mathcal{N} \overset{\mathcal{L}}{\models} \varphi$;
   (c) *Reduct Property.* If $L \subseteq L'$, $\mathcal{M}$ is a L'-structure of $\mathcal{L}$ and $\varphi \in \mathcal{L}[L]$, then $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi$ if and only if $\mathcal{M} \restriction L \overset{\mathcal{L}}{\models} \varphi$;
   (d) *Renaming Property.* Suppose that $r\colon L \to L'$ is a renaming. Then for each sentence $\varphi \in \mathcal{L}[L]$ there exists a sentence $\varphi^r \in \mathcal{L}[L]$ such that $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi$ if and only if $\mathcal{M}^r \overset{\mathcal{L}}{\models} \varphi^r$.

The field of abstract model theory originated with P. Lindsröm's landmark paper [12] and thrived during the 1970's. For an introduction to the subject we refer the reader to [2].

A logic $\mathcal{L}$ has *conjunctions* if for every pair of sentences $\varphi, \varphi' \in \mathcal{L}[L]$ there exists a sentence $\psi \in \mathcal{L}[L]$ such that

$$\mathcal{M} \overset{\mathcal{L}}{\models} \psi \qquad \text{if and only if} \qquad \mathcal{M} \overset{\mathcal{L}}{\models} \varphi \qquad \text{and} \qquad \mathcal{M} \overset{\mathcal{L}}{\models} \varphi'.$$

The logic $\mathcal{L}$ is said to have *negations* if for every sentence $\varphi \in \mathcal{L}[L]$ there exists a sentence $\psi \in \mathcal{L}[L]$ such that

$$\mathcal{M} \overset{\mathcal{L}}{\models} \psi \qquad \text{if and only if} \qquad \mathcal{M} \overset{\mathcal{L}}{\not\models} \varphi.$$

1.2. **Approximations.**

1.2. Definition. Let $\mathcal{L}$ be a logic. A *system of approximations* in $\mathcal{L}$ is a binary relation $\triangleleft$ on the sentences of $\mathcal{L}$ such that

(1) $\triangleleft$ is transitive;
(2) If $\varphi \triangleleft \varphi'$ and $\varphi \in \mathcal{L}[L]$, then $\varphi' \in \mathcal{L}[L]$;
(3) If $\varphi \triangleleft \varphi'$ and $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi$, then $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi'$.

If $\triangleleft$ is a system of approximations in a logic $\mathcal{L}$, φ is a sentence of $\mathcal{L}$ and $\varphi \triangleleft \varphi'$, we will say that φ' is a $\triangleleft$-*approximation* (or simply, an "approximation" if the underlying system of approximations is clear from the context) of φ. A *logic with approximations* is a pair $(\mathcal{L}, \triangleleft)$, where $\mathcal{L}$ is a logic and $\triangleleft$ is a system of approximations in $\mathcal{L}$.

If $(\mathcal{L}, \triangleleft)$ is a logic with approximations, $\mathcal{M}$ is a structure of $\mathcal{L}$, and φ is a sentence of $\mathcal{L}$, we will say that $\mathcal{M}$ *approximately satisfies* φ, and write $\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \varphi$, if $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi'$ for every $\triangleleft$-approximation φ' of φ.

1. Remarks.

(1) By condition (3) in Definition 1.2, the relation $\overset{\mathcal{L}}{\models}_{\mathcal{A}}$ is weaker than $\overset{\mathcal{L}}{\models}$.
(2) Every logic $\mathcal{L}$ can be regarded as a logic with approximations by defining $\triangleleft$ as the diagonal relation on the sentences of $\mathcal{L}$; in other words, the only approximation of each sentence is itself. We will refer to this system of approximations as the *discrete* system on $\mathcal{L}$. Notice that, relative to the discrete system, the relations $\overset{\mathcal{L}}{\models}$ and $\overset{\mathcal{L}}{\models}_{\mathcal{A}}$ are identical.

1.3. **Compactness.** A *theory* of a logic $\mathcal{L}$ is a set of sentences of $\mathcal{L}$. Let $(\mathcal{L}, \triangleleft)$ be a logic with approximations. We will say a theory Σ of $\mathcal{L}$ is *consistent* if there exists a structure $\mathcal{M}$ of $\mathcal{L}$ which approximately satisfies every sentence in Σ. We will say that Σ is *finitely consistent* if every finite subset of Σ is consistent.

We will say that a logic with approximations *satisfies the compactness theorem* if it has the property that every theory of $\mathcal{L}$ which is finitely consistent is consistent.

1.3. Remark. Our definition of satisfying the compactness theorem is more general than Lindström's definition in [12], since it involves approximate satisfaction rather than satisfaction.

1.4. **Elementary Chains.** Let $(\mathcal{L}, \triangleleft)$ be a logic with approximations. For an analytic structure $\mathcal{M}$, let $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{M})$ denote the set of sentences of $\mathcal{L}$ that are approximately satisfied by $\mathcal{M}$. If $\mathcal{N}$ is a structure of $\mathcal{L}$, we write $\mathcal{M} \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{N}$ to indicate that

$M \subseteq N$ and the structure $(\mathcal{N}, a)_{a\in M}$ approximately satisfies $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}((\mathcal{M}, a)_{a\in M})$. (Recall that the class of structures of a logic is assumed to be closed under expansions by constants.)

Let $(\mathcal{L}, \lhd)$ be a logic with approximations.. We will say that $(\mathcal{L}, \lhd)$ *satisfies the elementary chain property* if the following condition holds. Whenever

$$\mathcal{M}_0 \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{M}_1 \prec^{\mathcal{L}}_{\mathcal{A}} \dots \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{M}_n \prec^{\mathcal{L}}_{\mathcal{A}} \dots \qquad (n < \omega)$$

there exists a structure $\mathcal{M}$ of $\mathcal{L}$ such that $\mathcal{M}_n \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{M}$ for every $n < \omega$, and $\mathcal{M}$ is uniquely determined by $\bigcup_n \mathcal{M}$.

1.5. **Comparing Logics with Approximations.** Let $(\mathcal{L}, \lhd)$ and $(\mathcal{L}_1, \lhd_1)$ be logics with approximations such that $\mathcal{L}$ and $\mathcal{L}_1$ have the same structures. We will say that a sentence φ of $\mathcal{L}$ is *reducible* to $\mathcal{L}_1$ if the following condition holds. For every $\lhd$-approximation φ' of φ there exist two sentences $\psi[\varphi, \varphi']$ and $\psi'[\varphi, \varphi']$ of $\mathcal{L}_1$ such that:

(1) $\psi[\varphi, \varphi'] \lhd_1 \psi'[\varphi, \varphi']$;
(2) If $\mathcal{M}$ is a structure of $\mathcal{L}$,

$$\mathcal{M} \overset{\mathcal{L}}{\models} \varphi \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}_1}{\models} \psi[\varphi, \varphi'],$$
$$\mathcal{M} \overset{\mathcal{L}_1}{\models} \psi'[\varphi, \varphi'] \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}}{\models} \varphi'.$$

We will say that $(\mathcal{L}_1, \lhd_1)$ is an *extension* of $(\mathcal{L}, \lhd)$ if every sentence of $\mathcal{L}$ is reducible to $\mathcal{L}_1$. Two logics with approximations will be called *equivalent* if they are reducible to each other.

Intuitively, $(\mathcal{L}_1, \lhd_1)$ is an extension of $(\mathcal{L}, \lhd)$ if every sentence of $\mathcal{L}$ can be approximated by sentences of $\mathcal{L}'$. (Lemma 3.1, with $(\mathcal{H}, <)$ replaced by an arbitrary logic, shows that this intuition is indeed correct.) As a trivial but important example let us notice that if $(\mathcal{L}, \lhd)$ is a logic with approximations, $\mathcal{L}_1$ is a logic with the same structures as $\mathcal{L}$, every sentence of $\mathcal{L}$ is a sentence of $\mathcal{L}_1$, and $\overset{\mathcal{L}_1}{\models}$ extends $\overset{\mathcal{L}}{\models}$ (in the traditional mathematical sense of the word) then $\mathcal{L}_1$ with the discrete system of approximations (see Remark 1) is an extension of $(\mathcal{L}, \lhd)$.

1.6. **Weak Negation.** let $(\mathcal{L}, \lhd)$ be a logic with approximations. A *weak negation* on $(\mathcal{L}, \lhd)$ is a monadic operation $\overset{w}{\neg}$ on the sentences of $\mathcal{L}$ such that

(1) If $\varphi \in \mathcal{L}[L]$, then $\overset{w}{\neg}\varphi \in \mathcal{L}[L]$;
(2) If $\varphi \in \mathcal{L}[L]$ and $\mathcal{M}$ is an L-structure of $\mathcal{L}$, then

$$\mathcal{M} \overset{\mathcal{L}}{\models} \varphi \qquad \text{or} \qquad \mathcal{M} \models \overset{w}{\neg}\varphi;$$

(3) If φ' is an approximation of φ, then

$$\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \overset{w}{\neg}\varphi' \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}}{\not\models}_{\mathcal{A}} \varphi.$$

Note that $\mathcal{L}$ is a logic with negations and $\lhd$ is the discrete system of approximations of $\mathcal{L}$ (see Remark 1), then the negation of $\mathcal{L}$ is a weak negation on $(\mathcal{L}, \lhd)$.

If $(\mathcal{L}, \lhd)$ is a logic with approximations and $\mathcal{M}, \mathcal{N}$ are structures of $\mathcal{L}$, we write $\mathcal{M} \overset{\mathcal{L}}{\equiv}_{\mathcal{A}} \mathcal{N}$ if $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{M}) \subseteq \mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{N})$ and $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{N}) \subseteq \mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{M})$. Notice that if $(\mathcal{L}, \lhd)$ has a weak negation, then each of these inclusions implies the other.

2. The logic $\mathcal{H}$

We now state the properties of the logic $\mathcal{H}$ that will be invoked in the paper. The reader not entirely familiar with $\mathcal{H}$ may take the existence of a logic satisfying these properties as an axiom.

We shall refer to the structures of $\mathcal{H}$ as *analytic structures*.

2.1. Facts.

(1) The logic $\mathcal{H}$ has a system of approximations $<$.
(2) The pair $(\mathcal{H}, <)$ satisfies the compactness theorem and the elementary chain property, and has a weak negation, which is denoted neg( ).

The sentences of $\mathcal{H}$ are called *positive bounded sentences* and form a subclass of the class of first order sentences. The relation $\overset{\mathcal{H}}{\models}$ is the usual notion of satisfaction $\models$. The relations $\overset{\mathcal{H}}{\models}_{\mathcal{A}}$, $\overset{\mathcal{H}}{\prec}_{\mathcal{A}}$, and $\overset{\mathcal{H}}{\equiv}_{\mathcal{A}}$ will be denoted $\models_{\mathcal{A}}$, $\prec_{\mathcal{A}}$, and $\equiv_{\mathcal{A}}$, respectively.

3. The maximality of $\mathcal{H}$

Main Theorem. *Suppose that $(\mathcal{L}, \triangleleft)$ is a logic for analytic structures and that $(\mathcal{L}, \triangleleft)$*

- *extends $(\mathcal{H}, <)$,*
- *satisfies the compactness theorem,*
- *satisfies the elementary chain property, and*
- *has a weak negation.*

Then $(\mathcal{L}, \triangleleft)$ is equivalent to $(\mathcal{H}, <)$.

Let us now introduce some notation. For the rest of the the paper, $(\mathcal{L}, \triangleleft)$ will denote a logic with approximations that satisfies the hypothesis of the main theorem. We will prove a couple of lemmas about $(\mathcal{L}, \triangleleft)$ and then proceed to prove the theorem.

Since $(\mathcal{L}, \triangleleft)$ extends $(\mathcal{H}, <)$, for every pair of sentences σ, σ' of $\mathcal{H}$ with $\sigma < \sigma'$ there exists a pair of sentences $\psi[\sigma, \sigma']$ and $\psi'[\sigma, \sigma']$ of $\mathcal{L}$ with $\psi[\sigma, \sigma'] \triangleleft \varphi'[\sigma, \sigma']$ such that for every analytic structure $\mathcal{M}$,

$$\mathcal{M} \models \sigma \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}}{\models} \psi[\sigma, \sigma'],$$

$$\mathcal{M} \overset{\mathcal{L}}{\models} \psi'[\sigma, \sigma'] \qquad \text{implies} \qquad \mathcal{M} \models \sigma'.$$

For a theory Σ of $\mathcal{H}$, let

$$\Sigma^{\mathcal{L}} = \{\, \psi[\sigma', \sigma''] \mid \sigma' < \sigma'' \text{ and } \sigma < \sigma', \text{ for some } \sigma \in \Sigma\}.$$

The following lemma follows immediately from the definitions.

3.1. Lemma. *Let Σ be a theory of $\mathcal{H}$. Then for every analytic structure $\mathcal{M}$,*

$$\mathcal{M} \models_{\mathcal{A}} \Sigma \qquad \textit{if and only if} \qquad \mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \Sigma^{\mathcal{L}}.$$

3.2. Lemma. *Suppose that θ is a sentence of $\mathcal{L}$ that is not reducible to $\mathcal{H}$. Then there exist a $\triangleleft$-approximation θ' of θ and analytic structures $\mathcal{M}$ and $\mathcal{N}$ such that*

(1) $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$;

(2) $\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \theta$;

(3) $\mathcal{N} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \overset{w}{\neg}\theta'$.

Proof. Let Σ be the set of sentences σ of $\mathcal{H}$ such that for every analytic structure $\mathcal{K}$,

$$\mathcal{K} \overset{\mathcal{L}}{\models} \theta \qquad \text{implies} \qquad \mathcal{K} \models \sigma.$$

3.3. *Claim.* There exists a $\lhd$-approximation θ' of θ such that the theory

$$\Sigma^{\mathcal{L}} \cup \{\overset{w}{\neg}\theta'\}$$

is consistent.

Proof of Claim 3.3. Suppose that $\Sigma^{\mathcal{L}} \cup \{\overset{w}{\neg}\theta'\}$ is inconsistent for every $\lhd$-approximation θ' of θ, and fix one such $\lhd$-approximation θ'. Since $\mathcal{L}$ satisfies the compactness theorem, there exist sentences $\sigma_1, \ldots, \sigma_n \in \Sigma$ and approximations $\sigma_i < \sigma_i' < \sigma_i''$ for $i = 1, \ldots, n$ such that the theory

$$\{\, \psi_1[\sigma_1', \sigma_n''], \ldots, \psi_n[\sigma_n' \sigma_n''] \,\} \cup \{\overset{w}{\neg}\theta'\}$$

is inconsistent. We now show that for every analytic structure $\mathcal{K}$,

(*) $$\mathcal{K} \models \bigwedge_{i=1}^{n} \sigma_i' \qquad \text{implies} \qquad \mathcal{K} \overset{\mathcal{L}}{\models} \theta'.$$

Indeed, if $\mathcal{K} \models \bigwedge_i \sigma_i'$, then $\mathcal{K} \overset{\mathcal{L}}{\models} \psi[\sigma_i', \sigma_i'']$ for each i, so $\mathcal{K} \overset{\mathcal{L}}{\not\models}_{\mathcal{A}} \overset{w}{\neg}\theta'$. Therefore, $\mathcal{K} \overset{\mathcal{L}}{\not\models} \overset{w}{\neg}\theta'$ and hence, $\mathcal{K} \overset{\mathcal{L}}{\models} \theta'$ by (1) in the definition of weak negation.

By the definition of Σ, we have

(**) $$\mathcal{K} \overset{\mathcal{L}}{\models} \theta \qquad \text{implies} \qquad \mathcal{K} \models \bigwedge_{i=1}^{n} \sigma_i.$$

Since θ' is arbitrary, (*) and (**) show that θ is reducible to $\mathcal{H}$, which is contrary to our hypothesis, so Claim 3.3 is proved. □

Now take θ' as in Claim 3.3 and fix an analytic structure $\mathcal{N}$ such that

$$\mathcal{N} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \Sigma^{\mathcal{L}} \cup \{\overset{w}{\neg}\theta'\}.$$

By Lemma 3.1,

(***) $$\mathcal{N} \models_{\mathcal{A}} \Sigma.$$

3.4. *Claim.* $(\mathrm{Th}_{\mathcal{A}}(\mathcal{N}))^{\mathcal{L}} \cup \{\theta\}$ is consistent.

Proof of Claim 3.4. Suppose that $(\mathrm{Th}_{\mathcal{A}}(\mathcal{N}))^{\mathcal{L}} \cup \{\theta\}$ is inconsistent. Since $\mathcal{L}$ satisfies the compactness theorem, there exist $\sigma_1, \ldots, \sigma_n \in \mathrm{Th}_{\mathcal{A}}(\mathcal{N})$ and $\sigma_i < \sigma_i' < \sigma_i''$ for $i = 1, \ldots, n$ such that the theory

$$\{\, \psi[\sigma_1', \sigma_1''], \ldots, \psi[\sigma_n', \sigma_n''] \,\} \cup \{\theta\}$$

is inconsistent. Hence, if $\mathcal{K}$ is an analytic structure,

$$\mathcal{K} \overset{\mathcal{L}}{\models} \theta \qquad \text{implies} \qquad \mathcal{K} \not\models \bigwedge_{i=1}^{n} \sigma_i'.$$

Thus,

$$\mathcal{K} \overset{\mathcal{L}}{\models} \theta \qquad \text{implies} \qquad \mathcal{K} \models \operatorname{neg}\big(\bigwedge_{i=1}^{n} \sigma_i'\big),$$

so $\operatorname{neg}(\bigwedge_i \sigma_i') \in \Sigma$. By (***), we have $\mathcal{N} \models_{\mathcal{A}} \operatorname{neg}(\bigwedge_i \sigma_i')$. But (by (2) in the definition of weak negation) this contradicts the fact that $\mathcal{N} \models \bigwedge_i \sigma_i$. We have thus proved Claim 3.4. □

Now we can prove the lemma. Let $\mathcal{M}$ be an analytic structure such that $\mathcal{M} \models_{\mathcal{A}} (\mathrm{Th}_{\mathcal{A}}(\mathcal{N}))^{\mathcal{L}} \cup \{\theta\}$. By Lemma 3.1, $\mathcal{M} \models_{\mathcal{A}} \mathrm{Th}_{\mathcal{A}}(\mathcal{N})$, so $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$. By (***), the structures $\mathcal{M}$ and $\mathcal{N}$ are as required. □

3.5. Lemma. *Suppose that θ is a sentence of $\mathcal{L}$ that is not reducible to $\mathcal{H}$. Then there exist a $\triangleleft$-approximation θ' of θ and analytic structures $\mathcal{M}$ and $\mathcal{K}$ such that*

(1) $\mathcal{M} \prec_{\mathcal{A}} \mathcal{K}$;

(2) $\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \theta$;

(3) $\mathcal{K} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \overset{w}{\neg}\theta'$.

Proof. Let θ', $\mathcal{M}$, and $\mathcal{N}$ correspond to θ as in Lemma 3.2. By Lemma 3.1, it suffices to prove that the theory

$$\Sigma = (\mathrm{Th}_{\mathcal{A}}((\mathcal{M}, a)_{a\in M}))^{\mathcal{L}} \cup \{\overset{w}{\neg}\theta'\}$$

is consistent. But this follows from the assumption that $\mathcal{L}$ satisfies the compactness theorem, for every finite subset of Σ is approximately satisfied by a finite expansion of $\mathcal{N}$. □

3.6. Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be analytic structures such that $\mathcal{M} \prec_{\mathcal{A}} \mathcal{N}$. Then there exists an analytic structure $\mathcal{K}$ such that*

(1) $\mathcal{M} \prec_{\mathcal{A}}^{\mathcal{L}} \mathcal{K}$;

(2) $\mathcal{N} \prec_{\mathcal{A}} \mathcal{K}$.

Proof. By Lemma 3.1, it suffices to prove that the theory

$$\mathrm{Th}_{\mathcal{A}}^{\mathcal{L}}((\mathcal{M}, a)_{a\in M}) \cup (\mathrm{Th}_{\mathcal{A}}((\mathcal{N}, a, b)_{a\in M, b\in N\setminus M}))^{\mathcal{L}}$$

is consistent. We show that every finite subset of $(\mathrm{Th}_{\mathcal{A}}((\mathcal{N}, a, b)_{a\in M, b\in N\setminus M}))^{\mathcal{L}}$ is approximately satisfied by a finite expansion of $(\mathcal{M}, a)_{a\in M}$. This will prove the lemma.

Fix a subset of $(\mathrm{Th}_{\mathcal{A}}((\mathcal{N}, a, b)_{a\in M, b\in N\setminus M}))^{\mathcal{L}}$ of the form

$$\{\, \psi[\sigma_1', \sigma_1''], \dots, \psi[\sigma_n', \sigma_n''] \,\},$$

where there exist $\sigma_1, \dots, \sigma_n \in \mathrm{Th}_{\mathcal{A}}((\mathcal{M}, a)_{a\in M})$ such that $\sigma_i < \sigma_i' < \sigma_i''$ for $i = 1, \dots, n$. Let $\underline{a}_1, \dots \underline{a}_k$ and $\underline{b}_1, \dots \underline{b}_l$ be exhaustive lists of the names of the elements from M and $N \setminus M$ (respectively) that occur in the σ_i's. Since

$$(\mathcal{N}, a, b)_{a\in M, b\in N\setminus M} \models_{\mathcal{A}} \bigwedge_{i=1}^{n} \sigma_i(\underline{a}_1, \dots \underline{a}_k, \underline{b}_1, \dots \underline{b}_l),$$

there exist $c_1, \dots, c_l \in M$ such that

$$(\mathcal{M}, a_1, \dots a_k, c_1, \dots, c_l) \models \bigwedge_{i=1}^{n} \sigma_i'(\underline{a}_1, \dots \underline{a}_k, \underline{b}_1, \dots \underline{b}_l).$$

Hence,

$$(\mathcal{M}, a_1, \dots a_k, b_1, \dots b_k) \overset{\mathcal{L}}{\models} \psi[\sigma_i', \sigma_i''], \qquad \text{for } i = 1, \dots, n.$$

□

We now have all the material we need for the proof.

Proof of the main theorem. Suppose that $(\mathcal{L}, \triangleleft)$ is not equivalent to $(\mathcal{H}, <)$, and fix a sentence θ of $\mathcal{L}$ that is not reducible to $\mathcal{H}$. Lemma 3.5 provides a $\triangleleft$-approximation θ' of θ and analytic structures $\mathcal{M}_0$ and $\mathcal{M}_1$ such that

$$\mathcal{M}_0 \prec_{\mathcal{A}} \mathcal{M}_1, \qquad \mathcal{M}_0 \overset{\mathcal{L}}{\models}_{\mathcal{A}} \theta, \qquad \mathcal{M}_1 \overset{\mathcal{L}}{\models}_{\mathcal{A}} \overset{w}{\neg}\theta'.$$

Using Lemma 3.6 iteratively, we construct a chain

$$\mathcal{M}_0 \prec_{\mathcal{A}} \mathcal{M}_1 \prec_{\mathcal{A}} \dots \prec_{\mathcal{A}} \mathcal{M}_n \prec_{\mathcal{A}} \dots \qquad (n < \omega)$$

such that

$$\mathcal{M}_n \prec_{\mathcal{A}}^{\mathcal{L}} \mathcal{M}_{n+2}, \qquad \text{for } n < \omega.$$

Since $\mathcal{L}$ satisfies the elementary chain property, there exists a structure $\mathcal{M}$ of $\mathcal{L}$ such that $\mathcal{M}_n \prec_{\mathcal{A}}^{\mathcal{L}} \mathcal{M}$ for every n. In particular, $\mathcal{M}_0 \prec_{\mathcal{A}}^{\mathcal{L}} \mathcal{M}$ and $\mathcal{M}_1 \prec_{\mathcal{A}}^{\mathcal{L}} \mathcal{M}$; but then $\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \theta$ and $\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \overset{w}{\neg}\theta'$, which is impossible.

□

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