ANALYTIC STRUCTURES AND MODEL THEORETIC COMPACTNESS

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Introduction

In recent years, there has been considerable interest in replicating the successful development of first-order model theory in non first-order contexts. One such context that has emerged as an important test case is that of metric structures, i.e., structures whose sorts are metric spaces. The appeal of a rich model theory of metric structures lies in that, besides opening up new fields of interaction between model theory and other areas of mathematics (e.g., functional analysis and probability), it provides a natural generalization of the first-order theory; every structure of the of the type studied in first-order model theory can be seen as a metric structure where the underlying metric is discrete.

Several frameworks have been proposed to study metric structures from the perspective of model theory, to wit:

- · Henson's logic of *positive bounded formulas*
- · Ben-Yaacov's *compact abstract theories*, or *"cats"*
- · The framework of *continuous logic* developed by Ben-Yaacov and Usvyatsov.

Predecesssors include Chang and Keisler's *continuous model theory* [CK66] and Krivine's *real-valued logic* [Kri74].

Henson's logic of positive bounded formulas was developed in the mid 1970's, initially to understand the connections between Banach spaces and their ultrapowers (or nonstandard hulls). In the early papers [Hen74, Hen75, Hen76], Henson proved, among other things, versions of the compactness and Löwenheim-Skolem theorems for positive bounded formulas. This logical framework was used in various papers by Henson, Heinrich, and Moore that appeared during the 1970's and 1980's to study problems in Banach space geometry. (See, for example, [HHM83, HM83a, HM83b, HHM83, HH86, HHM86, HHM87].)

In the 1990's, in a series of papers [Iov94, Iov96, Iov97, Iov98, Iov99a, Iov99b, Iov99c], the author developed the theory of forking and stability for this language, showing that the resulting theory is surprisingly analogous to the first-order case. The emphasis on these papers was on structures of functional analysis whose sorts are Banach spaces, although it was observed that the basic apparatus could be adapted, with relatively straightforward adjustments, for the more general context of pointed metric spaces. A detailed introduction to the language of positive bounded formulas, via Banach space ultrapowers, appeared in [HI02]. This paper also includes an extensive bibliography.

Ben-Yaacov's concept of compact abstract classes, or "cats", originated in different settings; the concept is motivated by two situations: (i) existentially closed models of a universal first-order theory which not form an elementary class, and (ii) hyperimaginaries in a strictly simple first-order theory. Ben Yaacov showed that, in cats, positive formulas are powerful enough to yield not only model theoretic compactness, but also independence and the basic elements of simplicity theory. For a survey, the reader is referred to [BY05a].

The context of cats is more general than that of metric structures. However, Yen-Yaacov has shown that every cat that satisfies reasonable assumptions (namely, compactness) is metric, i.e., it comes from metric structures. See [BY05b].

The framework of continuous logic was proposed in the last few years Ben-Yaacov and Usvyatsov [BYU]. A self-contained introduction can be found in [BYBHU08]. The approach of continuous logic is similar to that of the logic of positive bounded formulas (in fact both approaches are mathematically equivalent), but its distinctive feature is that formulas are $[0,1]$-valued rather than $\{0,1\}$-valued; all continuous functions of $[0,1]^n$ into $[0,1]$ ($n \in \mathbb{N}$) are regarded as n-ary connectives, and the quantifiers of continuous logic are the inf and sup operators. An advantage of this approach over that of positive bounded formulas is that in some situations it is more natural to write functional equations (or inequations) involving compositions of real-valued continuous functions, suprema, and infima, rather than positive bounded axioms.

Continuous logic is a simplification of (and is equivalent to) the framework of *continuous model theory* proposed by Chang and Keisler in the 1960's. See [CK66].

All of these logics satisfy model theoretic compactness. Furthermore, all of them are equivalent in terms of expressive power.

In this communication we consider the following questions:

(1) Is it a mere coincidence that these logics are equivalent?
(2) Is here a more expressive logic that yields a powerful, compact model theory of metric structures?

We shall see that the answer to both questions is “no”. In fact, as will be shown below, in essence, what makes the aforementioned logics equivalent is the fact that they all satisfy a form of model-theoretic compactness.

1. Continuous logic

The result discussed in this communication is on abstract logics, and the model-theoretic frameworks mentioned in the introduction are examples of logics to which this result applies. As we introduce the concepts involved in the statement of the result, we will use continuous logic (CL hereafter) as the motivating example. However, aside from the fact that of the four proposals mentioned in the introduction CL is the most recent, the choice of CL as leading example in our context is rather arbitrary; as will be clear, the hypothesis of the main result are weak enough to be satisfied not only by many variations of the examples mentioned above, but also by classical extensions of first-order logic, e.g., extensions of first-order by quantifiers.

In this section we provide the basics of the syntax and semantics of CL. Previous familiarity with CL or any of the logics mentioned hereto is not needed to follow the ideas presented here. The reader interested in further aspects of continuous logic is referred to [BYU] or [BYBHU08]. (Either of these references can serve as self-contained introduction.)

The structures of CL are those of the form

$$\mathcal{M} = (\, M^{(s)}, d^{(s)}, R_i, F_j, a_k \mid s \in S, i \in I, j \in J, k \in K \,),$$

where

- $(\, M^{(s)}, d^{(s)} \mid s \in S \,)$ is a family of bounded metric spaces called the *sorts* of $\mathcal{M}$; the metrics $d^{(s)}$ will be called the metrics of $\mathcal{M}$
- For each $i \in I$, R_i is a uniformly continuous function of the form

  $$R_i \colon M^{(s_1)} \times \cdots \times M^{(s_n)} \to \mathbb{R},$$

  where n is an integer $s_1, \ldots, s_n \in S$; the functions R_i are called the *predicates* of $\mathcal{M}$
- For each $j \in J$, F_j is a uniformly continuous function of the form

  $$F_j \colon M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)},$$

  where n is an integer $s_0, \ldots, s_n \in S$; the functions F_j are called the *operations* of $\mathcal{M}$
- For each $k \in K$, a_k is a distinguished element of one of the sorts of $\mathcal{M}$; the elements a_k are called the *constants* of $\mathcal{M}$.

These will be called *metric structures.* The restriction that the sorts be bounded is given to facilitate the syntax and does not limit the class of structures under consideration; indeed, if (M, d) is an unbounded metric space and $a \in M$, (M, d) may be replaced by sorts (M_n, d), where $M_n = \{\, x \in M \mid d(x, a) \le n \,\}$.

Examples of metric structures abound in classical mathematics: metric spaces, Banach spaces, operator spaces, measure algebras, and the kinds of structures traditionally studied in model theory; in this last case, the unmentioned metrics are regarded as discrete. More examples can be found in [BYBHU08], in [BYU], and in [HI02], which focuses on structures based on Banach spaces.

Let

$$\mathcal{M} = (\, M^{(s)}, d^{(s)}, R_i, F_j, a_k \mid s \in S, i \in I, j \in J, k \in K \,),$$

be a metric structure. A *signature* for $\mathcal{M}$ includes:

- A distinct binary relation symbol for each metric of $\mathcal{M}$
- A distinct n-ary relation symbol for each n-ary predicate of $\mathcal{M}$
- A distinct n-ary function symbol for each n-ary function of $\mathcal{M}$
- A distinct constant symbol for each constant of $\mathcal{M}$
- A *modulus of uniform continuity* for each of these functions; a modulus of uniform continuity for a function is a map $\epsilon \mapsto \delta$ such that whenever two arguments are, variable by variable, within distance δ, then their images are within distance ϵ. If L is a signature for $\mathcal{M}$ we say that $\mathcal{M}$ is and *L-structure.*

Since the sorts of $\mathcal{M}$ are bounded and the predicates of $\mathcal{M}$ are uniformly continuous, the range of each predicate is a bounded subset of $\mathbb{R}$. Without loss of generality, we can assume that the diameter of each sort of $\mathcal{M}$ is at most 1, and that the range of each predicate of $\mathcal{M}$ is a subset of $[0, 1]$.

Let $\mathcal{M}$ be an L-structure. To simplify the exposition, we will assume that $\mathcal{M}$ has a single sort, (M, d). Also, to ease notation, we will use the same symbol to denote any metric, relation, function, or constant of $\mathcal{M}$ and its respective metric, relation, function, or constant symbol in L.

The terms of L are defined as in ordinary first-order logic (i.e., starting with an infinite set of variables and and iterating function symbols). The *real-valued formulas* of L are defined inductively as follows:

- All the expressions of the form $d(t_1, t_2)$, where t_1, t_2 are terms L, and all the expressions of the form $R(t_1, \dots, t_n)$, where R is an n-ary predicate symbol and $t_1, \dots, t_n$ are terms of L, are real-valued formulas of L.
- If $\varphi_1, \dots, \varphi_n$ are real-valued formulas of L and $c\colon [0,1]^n \to [0,1]$ is a continuous function, then $c(\varphi_1, \dots, \varphi_n)$ is a real-valued formula of L.
- If x is a variable of L and φ is a real-valued formula of L, the expressions $\sup_x \varphi$ and $\inf_x \varphi$ are real-valued formulas of L.

The analogy with first-order logic should be clear. In CL, formulas represent $[0,1]$-valued instead of $\{0,1\}$-valued functions; the n-ary connectives of CL are all the continuous functions of the form $c\colon [0,1]^n \to [0,1]$, and the quantifiers are the inf and sup operators.

For every real-valued formula of φ of L there exists a nonnegative integer n such φ such that for every L-structure $\mathcal{M}$ as above, φ naturally defines a function $\varphi^M\colon M^n \to [0,1]$. The formula φ is said to be a *sentence* if $n = 0$.

1.1. **Definition.** Let $\mathcal{M}$ and $\mathcal{N}$ be L structures.

(1) We say that $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$, if $\varphi^M = \varphi^N$ for every sentence φ of L.
(2) If $\mathcal{M}$ is a substructure of $\mathcal{N}$, we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ if the structures $(\mathcal{M}, a \mid a \in M)$ and $(\mathcal{N}, a \mid a \in M)$ are elementarily equivalent.

The following is a basic property of CL. The proof can be found in [BYBHU08] and [BYU].

1.2. **Theorem.** *Suppose that $(I, <)$ is a linearly ordered set and $(\,\mathcal{M}_i \mid i \in I\,)$ is a family of L-structures such that $\mathcal{M}_i \prec \mathcal{M}_j$ whenever $< j$. Then, for every $i \in I$, $\mathcal{M}_i \prec \bigcup_{i\in I} \mathcal{M}_i$.*

1.3. **Definition.** If $\mathcal{M}$ is a metric structure, let $(\mathcal{M}, \mathbb{R}, \leq)$ denote the structure that includes, in addition to the structure already present in $\mathcal{M}$, the set $\mathbb{R}$ as a distinguished sort, and the order $\leq$ on $\mathbb{R}$. An *L-inequality* is an expression of the form $\varphi \leq r$ or $\varphi \geq r$, where φ is a real-valued formula of L and $r \in \mathbb{R}$.

Clearly, $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ satisfy the same inequalities.

2. Abstract Logics and Approximations

If L and L' are multi-sorted signatures, a *renaming* is a bijection $r\colon L \to L'$ that maps sort symbols onto sort symbols, relation symbols onto relation symbols, and function symbols onto function symbols, respecting. If $r\colon L \to L'$ is a renaming and $\mathcal{M}$ is an L-structure, $\mathcal{M}^r$ denotes the structure that results from converting $\mathcal{M}$ into an L'-structure through the map r. The structure $\mathcal{M}^r$ is called a renaming of $\mathcal{M}$.

Let us recall Lindstrom's definition of abstract logic [Lin69]:

2.1. **Definition.** A logic $\mathcal{L}$ consists of the following items.

(1) A class of structures, called the *structures of* $\mathcal{L}$, that is closed under isomorphisms, renamings, expansion by constants, and reducts.

(2) For each multi-sorted signature L, a set $\mathcal{L}[L]$ called the *L-sentences of* $\mathcal{L}$, such that $\mathcal{L}[L] \subseteq \mathcal{L}[L']$ when $L \subseteq L'$.

(3) A binary relation $\stackrel{\mathcal{L}}{\models}$ between structures and sentences of $\mathcal{L}$ such that:

  (a) If $\mathcal{M}$ is an L-structure of $\mathcal{L}$ and $\mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi$, then $\varphi \in \mathcal{L}[L]$.

  (b) *Isomorphism Property.* If $\mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi$ and $\mathcal{M}$ is isomorphic to $\mathcal{N}$, then $\mathcal{N} \stackrel{\mathcal{L}}{\models} \varphi$;

  (c) *Reduct Property.* If $L \subseteq L'$, $\mathcal{M}$ is a L'-structure of $\mathcal{L}$ and $\varphi \in \mathcal{L}[L]$, then $\mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi$ if and only if $\mathcal{M} \restriction L \stackrel{\mathcal{L}}{\models} \varphi$;

  (d) *Renaming Property.* Suppose that $r\colon L \to L'$ is a renaming. Then for each sentence $\varphi \in \mathcal{L}[L]$ there exists a sentence $\varphi^r \in \mathcal{L}[L]$ such that $\mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi$ if and only if $\mathcal{M}^r \stackrel{\mathcal{L}}{\models} \varphi^r$.

A logic $\mathcal{L}$ has *conjunctions* if for every pair of sentences $\varphi, \varphi' \in \mathcal{L}[L]$ there exists a sentence $\psi \in \mathcal{L}[L]$ such that

$$\mathcal{M} \stackrel{\mathcal{L}}{\models} \psi \qquad \text{if and only if} \qquad \mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi \qquad \text{and} \qquad \mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi'.$$

The logic $\mathcal{L}$ is said to have *negations* if for every sentence $\varphi \in \mathcal{L}[L]$ there exists a sentence $\psi \in \mathcal{L}[L]$ such that

$$\mathcal{M} \stackrel{\mathcal{L}}{\models} \psi \qquad \text{if and only if} \qquad \mathcal{M} \stackrel{\mathcal{L}}{\not\models} \varphi.$$

We now turn focus on model theoretic compactness. All the model theoretic frameworks for metric structures mentioned in the introduction satisfy a form of the compactness theorem. However, the form compactness satisfied by these logics is not a literal translation of the classical compactness theorem of first-order logic; the reason is that those logics that do not have negations in the sense defined above. The statement of compactness for those more general contexts involves topological perturbations. The compactness of first-order logic is a particular case when the topologies involved in the perturbations are discrete. This is a peculiarity of first-order due to the fact that its space of truth-values, $\{0, 1\}$, is a discrete. In [Iov01], the author introduced a notion *approximations* sentences for abstract logics that captures the kinds of perturbations needed to state model theoretic compactness for logics with truth values that are not necessarily discrete. We now recall the concept of approximation introduced in [Iov01]:

2.2. **Definition.** Let $\mathcal{L}$ be a logic. A *system of approximations* in $\mathcal{L}$ is a binary relation $\lhd$ on the sentences of $\mathcal{L}$ such that

(1) $\lhd$ is transitive;
(2) If $\varphi \lhd \varphi'$ and $\varphi \in \mathcal{L}[L]$, then $\varphi' \in \mathcal{L}[L]$;
(3) If $\varphi \lhd \varphi'$ and $\mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi$, then $\mathcal{M} \stackrel{\mathcal{L}}{\models} \varphi'$.

If $\lhd$ is a system of approximations in a logic $\mathcal{L}$, φ is a sentence of $\mathcal{L}$ and $\varphi \lhd \varphi'$, we will say that φ' is a *$\lhd$-approximation* (or simply, an "approximation", if the underlying system of approximations is clear from the context) of φ. A *logic with approximations* is a pair $(\mathcal{L}, \lhd)$, where $\mathcal{L}$ is a logic and $\lhd$ is a system of approximations in $\mathcal{L}$.

If $(\mathcal{L}, \lhd)$ is a logic with approximations, $\mathcal{M}$ is a structure of $\mathcal{L}$, and φ is a sentence of $\mathcal{L}$, we will say that $\mathcal{M}$ *approximately satisfies* φ, and write $\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \varphi$, if $\mathcal{M} \overset{\mathcal{L}}{\models} \varphi'$ for every $\lhd$-approximation φ' of φ.

2.3. **Remarks.**

(1) By condition (3) in Definition 2.2, the relation $\overset{\mathcal{L}}{\models}_{\mathcal{A}}$ is weaker than $\overset{\mathcal{L}}{\models}$.
(2) Every logic $\mathcal{L}$ can be regarded naturally as a logic with approximations by defining $\lhd$ as the diagonal relation on the sentences of $\mathcal{L}$; in other words, the only approximation of each sentence is itself. We will refer to this system of approximations as the *discrete* system on $\mathcal{L}$. Notice that, relative to the discrete system, the relations $\overset{\mathcal{L}}{\models}$ and $\overset{\mathcal{L}}{\models}_{\mathcal{A}}$ are identical.

A *theory* of a logic $\mathcal{L}$ is a set of sentences of $\mathcal{L}$. Let $(\mathcal{L}, \lhd)$ be a logic with approximations. We will say a theory Σ of $\mathcal{L}$ is *consistent* if there exists a structure $\mathcal{M}$ of $\mathcal{L}$ which approximately satisfies every sentence in Σ. We will say that Σ is *finitely consistent* if every finite subset of Σ is consistent.

2.4. **Definition.** Let $(\mathcal{L}, \lhd)$ be a logic with approximations. We will say that $(\mathcal{L}, \lhd)$ *satisfies the compactness theorem* if it has the property that every theory of $\mathcal{L}$ which is finitely consistent is consistent.

Let $(\mathcal{L}, \lhd)$ be a logic with approximations. If $\mathcal{M}$ is a structure of $\mathcal{L}$, we denote by $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{M})$ the set of sentences of $\mathcal{L}$ that are approximately satisfied by $\mathcal{M}$. If $\mathcal{N}$ is a structure of $\mathcal{L}$, we write $\mathcal{M} \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{N}$ to indicate that $M \subseteq N$ and the structure $(\mathcal{N}, a)_{a\in M}$ approximately satisfies $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}((\mathcal{M}, a)_{a\in M})$. (Recall that the class of structures of a logic is assumed to be closed under expansions by constants.)

2.5. **Definition.** Let $(\mathcal{L}, \lhd)$ be a logic with approximations. We will say that $(\mathcal{L}, \lhd)$ *satisfies the elementary chain property* if the following condition holds. Whenever

$$\mathcal{M}_0 \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{M}_1 \prec^{\mathcal{L}}_{\mathcal{A}} \dots \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{M}_n \prec^{\mathcal{L}}_{\mathcal{A}} \dots \qquad (n < \omega)$$

there exists a structure $\mathcal{M}$ of $\mathcal{L}$ such that $\mathcal{M}_n \prec^{\mathcal{L}}_{\mathcal{A}} \mathcal{M}$ for every $n < \omega$, and $\mathcal{M}$ is uniquely determined by $\bigcup_n \mathcal{M}$.

3. Continuous Logic as an Abstract Logic

In this section we state the properties of CL that, as Theorem 4.1 shows, characterize it.

(1) The class of structures of CL is the class of all metric structures.
(2) The L-sentences of CL, for a given signature L, are all the positive boolean combinations of inequalities $\varphi \leq r$ or $\varphi \geq r$, where φ is a real-valued sentence of L and $r \in \mathbb{R}$; see Definition 1.3.
(3) The relation $\overset{\mathrm{CL}}{\models}$ is the obvious one: $\mathcal{M} \overset{\mathrm{CL}}{\models} \varphi \leq r$ iff $\varphi^{\mathcal{M}} \leq r$, and $\mathcal{M} \overset{\mathrm{CL}}{\models} \varphi \geq r$ iff $\varphi^{\mathcal{M}} \geq r$
(4) CL has approximations: the approximations of the inequality $\varphi \leq r$ are the inequalities of the form $\varphi \leq s$, where $s > r$, and the approximations of $\varphi \geq r$ are the inequalities of the form $\varphi \geq s$, where $s < r$; if $\sigma_1, \dots, \sigma_n$ are inequalities and $B(\sigma_1, \dots, \sigma_n)$ is a positive boolean combination of $\sigma_1, \dots, \sigma_n$, than the approximations of $B(\sigma_1, \dots, \sigma_n)$ are all the

expressions of the form $B(\tau_1, \dots, \tau_n)$, where τ_i is an approximation of σ_i, for $i = 1, \dots, n$. We will write $\sigma < \tau$ if τ is an approximation of σ.
(5) The pair $(\mathrm{CL}, <)$ satisfies the compactness theorem and the elementary chain property; see Definitions 2.4 and 2.5.
(6) The pair $(\mathrm{CL}, <)$ has a weak negation: define $\overset{w}{\neg}(\varphi \le r) = \varphi \ge r$, $\overset{w}{\neg}(\varphi \ge r) = \varphi \le r$, and, inductively, $\overset{w}{\neg}(\sigma \wedge \tau) = \overset{w}{\neg}(\sigma) \vee \overset{w}{\neg}(\tau)$ and $\overset{w}{\neg}(\sigma \vee \tau) = \overset{w}{\neg}(\sigma) \wedge \overset{w}{\neg}(\tau)$.

4. The maximality of Continuous Logic

Let $(\mathcal{L}, \lhd)$ and $(\mathcal{L}_1, \lhd_1)$ be logics with approximations such that $\mathcal{L}$ and $\mathcal{L}_1$ have the same structures. We will say that a sentence φ of $\mathcal{L}$ is *reducible* to $\mathcal{L}_1$ if the following condition holds. For every $\lhd$-approximation φ' of φ there exist two sentences $\psi[\varphi, \varphi']$ and $\psi'[\varphi, \varphi']$ of $\mathcal{L}_1$ such that:

(1) $\psi[\varphi, \varphi'] \lhd_1 \psi'[\varphi, \varphi']$;
(2) If $\mathcal{M}$ is a structure of $\mathcal{L}$,

$$\mathcal{M} \overset{\mathcal{L}}{\models} \varphi \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}_1}{\models} \psi[\varphi, \varphi'],$$
$$\mathcal{M} \overset{\mathcal{L}_1}{\models} \psi'[\varphi, \varphi'] \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}}{\models} \varphi'.$$

We will say that $(\mathcal{L}_1, \lhd_1)$ is an *extension* of $(\mathcal{L}, \lhd)$ if every sentence of $\mathcal{L}$ is reducible to $\mathcal{L}_1$. Two logics with approximations will be called *equivalent* if they are reducible to each other.

Intuitively, $(\mathcal{L}_1, \lhd_1)$ is an extension of $(\mathcal{L}, \lhd)$ if every sentence of $\mathcal{L}$ can be approximated by sentences of $\mathcal{L}'$. As a trivial but important example let us notice that if $(\mathcal{L}, \lhd)$ is a logic with approximations, $\mathcal{L}_1$ is a logic with the same structures as $\mathcal{L}$, every sentence of $\mathcal{L}$ is a sentence of $\mathcal{L}_1$, and $\overset{\mathcal{L}_1}{\models}$ extends $\overset{\mathcal{L}}{\models}$ (in the traditional mathematical sense of the word) then $\mathcal{L}_1$ paired with the discrete system of approximations (see Remark 2.3) is an extension of $(\mathcal{L}, \lhd)$.

let $(\mathcal{L}, \lhd)$ be a logic with approximations. A *weak negation* on $(\mathcal{L}, \lhd)$ is a monadic operation $\overset{w}{\neg}$ on the sentences of $\mathcal{L}$ such that

(1) If $\varphi \in \mathcal{L}[L]$, then $\overset{w}{\neg}\varphi \in \mathcal{L}[L]$;
(2) If $\varphi \in \mathcal{L}[L]$ and $\mathcal{M}$ is an L-structure of $\mathcal{L}$, then

$$\mathcal{M} \overset{\mathcal{L}}{\models} \varphi \qquad \text{or} \qquad \mathcal{M} \models \overset{w}{\neg}\varphi;$$

(3) If φ' is an approximation of φ, then

$$\mathcal{M} \overset{\mathcal{L}}{\models}_{\mathcal{A}} \overset{w}{\neg}\varphi' \qquad \text{implies} \qquad \mathcal{M} \overset{\mathcal{L}}{\not\models}_{\mathcal{A}} \varphi.$$

Note that if $\mathcal{L}$ is a logic with negations and $\lhd$ is the discrete system of approximations of $\mathcal{L}$ (see Remark 2.3), then the negation of $\mathcal{L}$ is a weak negation on $(\mathcal{L}, \lhd)$.

If $(\mathcal{L}, \lhd)$ is a logic with approximations and $\mathcal{M}, \mathcal{N}$ are structures of $\mathcal{L}$, we write $\mathcal{M} \overset{\mathcal{L}}{\equiv}_{\mathcal{A}} \mathcal{N}$ if $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{M}) \subseteq \mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{N})$ and $\mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{N}) \subseteq \mathrm{Th}^{\mathcal{L}}_{\mathcal{A}}(\mathcal{M})$. Notice that if $(\mathcal{L}, \lhd)$ has a weak negation, then each of these inclusions implies the other.

The following result was proved in [Iov01]:

4.1. **Theorem.** *Suppose that $(\mathcal{L}, \lhd)$ is a logic for metric structures and that $(\mathcal{L}, \lhd)$*

- *extends* (CL, $<$),
- *satisfies the compactness theorem,*
- *satisfies the elementary chain property, and*
- *has a weak negation.*

Then $(\mathcal{L}, \lhd)$ *is equivalent to* (CL, $<$).

This theorem shows, among other things, the equivalence among all the logics for metric structures mentioned here. For instance, let us show the equivalence between CL and the logic $\mathcal{L}_{\mathrm{PB}}$ of positive bounded formulas. Both logics satisfy the compactness theorem and the elementary chain property (the proofs can be found in any basic expositions available for these two logics; see, for example, [BYBHU08] for the CL and [HI02] for $\mathcal{L}_{\mathrm{PB}}$), so we just have to note that CL is an extension of $\mathcal{L}_{\mathrm{PB}}$ in the sense defined at the beginning of this section. Recall from Section 1 that two metric structures $\mathcal{M}, \mathcal{N}$ are elementarily equivalent in CL if and only if the structures $(\mathcal{M}, \mathbb{R}, <)$ and $(\mathcal{N}, \mathbb{R}, <)$ satisfy the same inequalities of the form $\varphi \leq r$ and $\varphi \geq r$, where $r \in \mathbb{R}$ and φ is a $[0,1]$-valued function built up from the ($[0,1]$-valued) predicates of L by using continuous functions $c\colon [0,1]^n \to [0,1]$ as connectives and the operators sup and inf as quantifiers. The positive bounded formulas of a signature L are all the expressions that can be built up from the basic inequalities of the form $R(t_1, \dots, t_n) \leq r$ and $R(t_1, \dots, t_n) \geq r$, where $r \in \mathbb{R}$, $t_1, \dots, t_n$ are terms of L and R is a predicate of L, by using the connectives $\wedge, \vee$ and the first-order the quantifiers $\exists, \forall$ (which range over the bounded sorts — hence the name "positive bounded"). If ψ is a positive bounded formula, the approximations of ψ are defined as the formulas that result from "relaxing" all the estimates that occur in ψ, i.e., replacing all the inequalities of the form $R(t_1, \dots, t_n) \leq r$ that occur in ψ by inequalities of the form $R(t_1, \dots, t_n) \leq s$ for some $s > r$ and, similarly, replacing all the inequalities of the form $R(t_1, \dots, t_n) \geq r$ by $R(t_1, \dots, t_n) \geq s$ for some $s < r$. It is easy to see, by induction on the complexity of formulas, that every positive bounded formula is reducible to CL: for the quantifier-free case, it suffices to observe that every formula of the form

$$\bigwedge_{1 \leq i \leq N} \Big(\bigvee_{1 \leq \alpha \leq m(i)} R_{1,\alpha}(\bar{t}) \leq r_{1,\alpha} \vee \bigvee_{1 \leq \beta \leq n(i)} R_{2,\beta}(\bar{t}) \geq r_{2,\beta} \Big),$$

where $R_{1,\alpha}, R_{2,\beta}$ are $[0,1]$-valued predicates and $r_{1,\alpha}, r_{2,\beta} \in [0,1]$ for $1 \leq \alpha \leq m(i), 1 \leq \beta \leq n(i)$, is equivalent to the inequality

$$\max_{1 \leq i \leq N} \Big[\min(\min_{1 \leq \alpha \leq m(i)} R_{1,\alpha}(\bar{t}) \dot{-} r_{1,\alpha}, \min_{1 \leq \beta \leq n(i)} r_{2,\beta} \dot{-} R_{2,\beta}(\bar{t})) \Big] \leq 0,$$

where $\dot{-}$ denotes the truncated difference on $[0,1]$, i.e., $x \dot{-} y$ is $x - y$ if $x \geq y$ and 0 if $x < y$; the truncated difference is a continuous function from $[0,1]^2$ into $[0,1]$, and hence a binary connective of CL, so the preceding inequality is of the type defined in Definition 1.3. The quantifier step of the induction is given by the fact for every real-valued function f and every $\epsilon > 0$,

- $\exists x (f(x) \leq r) \Longrightarrow \inf_x f(x) \leq r \Longrightarrow \exists x (f(x) \leq r + \epsilon)$
- $\exists x (f(x) \geq r) \Longrightarrow \sup_x f(x) \geq r \Longrightarrow \exists x (f(x) \geq r - \epsilon)$
- $\forall x (f(x) \leq r) \Longleftrightarrow \sup_x f(x) \leq r$
- $\forall x (f(x) \geq r) \Longleftrightarrow \inf_x f(x) \geq r$.

The continuous model theory framework of Chang and Keisler [CK66] extends CL (the main difference is that Chang and Keisler include many more quantifiers);

but then, since both logics satisfy the compactness theorem and the elementary chain property, they must be equivalent.

First-order logic extends CL, since every inequality of the type defined in Definition 1.3 is a first-order sentence. The extension is proper because, as a logic for metric structures, first-order does not satisfy the compactness theorem. (In fact, the expressive power of first-order logic on Banach spaces is known to be quite high [SS78].) It seems rather striking to us that there is no logic strictly between CL and first-order satisfying the conditions of Theorem 4.1.

References

[BY05a] Itay Ben-Yaacov, *Compactness and independence in non first order frameworks*, Bull. Symbolic Logic **11** (2005), no. 1, 28–50. MR MR2125148 (2006d:03054)

[BY05b] ———, *Uncountable dense categoricity in cats*, J. Symbolic Logic **70** (2005), no. 3, 829–860. MR MR2155268 (2006g:03063)

[BYBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427. MR MR2436146

[BYU] Itay Ben-Yaacov and Alex Usvyatsov, *Continuous first order logic and local stability*, To appear in the Transactions of the American Mathematical Society.

[CK66] Chen-chung Chang and H. Jerome Keisler, *Continuous model theory*, Annals of Mathematics Studies, No. 58, Princeton Univ. Press, Princeton, N.J., 1966. MR MR0231708 (38 #36)

[Hen74] C. W. Henson, *The isomorphism property in nonstandard analysis and its use in the theory of Banach spaces*, J. Symbolic Logic **39** (1974), 717–731. MR 50 #12713

[Hen75] ———, *When do two Banach spaces have isometrically isomorphic nonstandard hulls?*, Israel J. Math. **22** (1975), no. 1, 57–67.

[Hen76] ———, *Nonstandard hulls of Banach spaces*, Israel J. Math. **25** (1976), no. 1-2, 108–144.

[HH86] Stefan Heinrich and C. Ward Henson, *Banach space model theory. II. Isomorphic equivalence*, Math. Nachr. **125** (1986), 301–317.

[HHM83] S. Heinrich, C. W. Henson, and L. C. Moore, Jr., *Elementary equivalence of L_1-preduals*, Banach space theory and its applications (Bucharest, 1981), Springer, Berlin, 1983, pp. 79–90. MR 84j:46021

[HHM86] ———, *Elementary equivalence of $C_\sigma(K)$ spaces for totally disconnected, compact Hausdorff K*, J. Symbolic Logic **51** (1986), no. 1, 135–146. MR 87f:03097

[HHM87] ———, *A note on elementary equivalence of $C(K)$ spaces*, J. Symbolic Logic **52** (1987), no. 2, 368–373. MR 88k:03078

[HI02] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110. MR 1 967 834

[HM83a] C. W. Henson and L. C. Moore, Jr., *The Banach spaces $l_p(n)$ for large p and n*, Manuscripta Math. **44** (1983), no. 1-3, 1–33. MR 84g:46026

[HM83b] ———, *Nonstandard analysis and the theory of Banach spaces*, Nonstandard analysis—recent developments (Victoria, B.C., 1980), Springer, Berlin, 1983, pp. 27–112. MR 85f:46033

[Iov94] José Iovino, *Stable theories in functional analysis*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 1994.

[Iov96] ———, *The Morley rank of a Banach space*, J. Symbolic Logic **61** (1996), no. 3, 928–941. MR 97j:03072

[Iov97] ———, *Definability in functional analysis*, J. Symbolic Logic **62** (1997), no. 2, 493–505. MR 98i:03046

[Iov98] ———, *Types on stable Banach spaces*, Fund. Math. **157** (1998), no. 1, 85–95. MR 99d:46013

[Iov99a] ———, *Stable Banach spaces and Banach space structures. I. Fundamentals*, Models, algebras, and proofs (Bogotá, 1995), Dekker, New York, 1999, pp. 77–95. MR 2000h:03064a

[Iov99b] ———, *Stable Banach spaces and Banach space structures. II. Forking and compact topologies*, Models, algebras, and proofs (Bogotá, 1995), Dekker, New York, 1999, pp. 97–117. MR 2000h:03064b

[Iov99c] ———, *Stable models and reflexive Banach spaces*, J. Symbolic Logic **64** (1999), no. 4, 1595–1600. MR 2001e:03068

[Iov01] ———, *On the maximality of logics with approximations.*, J. Symbolic Logic **66** (2001), no. 4, 1909–1918.

[Kri74] J.-L. Krivine, *Langages à valeurs réelles et applications*, Fund. Math. **81** (1974), 213–253, Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, III. MR 50 #1873

[Lin69] P. Lindström, *On extensions of elementary logic*, Theoria **35** (1969), 1–11. MR 39 #5330

[SS78] S. Shelah and J. Stern, *The Hanf number of the first order theory of Banach spaces*, Trans. Amer. Math. Soc. **244** (1978), 147–171.

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